

The Hamiltonian Syllogistic

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Abstract

This paper undertakes a re-examination of Sir William Hamilton's doctrine of the *quantification of the predicate*. Hamilton's doctrine comprises two theses. First, the predicates of traditional syllogistic sentence-forms contain implicit existential quantifiers, so that, for example, *All p are q* is to be understood as *All p are some q*. Second, these implicit quantifiers can be meaningfully dualized to yield novel sentence-forms, such as, for example, *All p are all q*. Hamilton attempted to provide a deductive system for his language, along the lines of the classical syllogisms. We show, using techniques unavailable to Hamilton, that such a system does exist, though with qualifications that distinguish it from its classical counterpart.

1 Introduction

By the *classical syllogistic*, we understand the set of English sentences of the forms

$$\begin{array}{ll} \text{Every } p \text{ is a } q & \text{Some } p \text{ is a } q \\ \text{No } p \text{ is a } q & \text{Some } p \text{ is not a } q, \end{array} \quad (1)$$

where p and q are common (count) nouns. These sentence-forms are evidently logically equivalent to the following more cumbersome forms:

$$\begin{array}{ll} \text{Every } p \text{ is identical to some } q & \text{Some } p \text{ is identical to some } q \\ \text{No } p \text{ is identical to any } q & \text{Some } p \text{ is not identical to any } q. \end{array} \quad (2)$$

By the *Hamiltonian syllogistic*, we understand the set of sentences of the forms (2), together with the set of sentences of the forms

$$\begin{array}{ll} \text{Every } p \text{ is identical to every } q & \text{Some } p \text{ is identical to every } q \\ \text{No } p \text{ is identical to every } q & \text{Some } p \text{ is not identical to every } q, \end{array} \quad (3)$$

obtained from (2) by dualizing the second quantifier. Taking account of the equivalence of the forms (1) and (2), we informally regard the classical syllogistic as a subset of the Hamiltonian syllogistic. The sentence-forms (3) have no idiomatic English equivalents. We take their respective first-order translations to be

$$\begin{array}{ll} \forall x(p(x) \rightarrow \forall y(q(y) \rightarrow x = y)) & \exists x(p(x) \wedge \forall y(q(y) \rightarrow x = y)) \\ \forall x(p(x) \rightarrow \neg \forall y(q(y) \rightarrow x = y)) & \exists x(p(x) \wedge \neg \forall y(q(y) \rightarrow x = y)). \end{array} \quad (4)$$

Thus, for example, *Every p is identical to every q* is true just in case there are no *ps*, or no *qs*, or there is exactly one *p* and one *q*, and they are identical. Observe that determiners in subjects are taken to outscope those in predicates. Observe also that verb negation is taken to outscope the following predicate determiner. Thus, for example, *Some p is not identical to every q* is true just in case some *p* is distinct from some *q*.

By a *classical syllogism* we understand any of the valid two-premise argument patterns for sentences of the forms (1), for example:

$$\begin{array}{ll} \text{Every } p \text{ is a } q & \text{Every } q \text{ is an } r \\ \hline \text{Every } q \text{ is an } r & \text{Some } p \text{ is a } q \\ \hline \text{Every } p \text{ is an } r & \text{Some } p \text{ is an } r. \end{array}$$

It is known that the classical syllogisms—with one or two minor additions—constitute a sound and complete proof system for the classical syllogistic. Does there exist a comparable system of rules for the Hamiltonian syllogistic?

The Hamiltonian syllogistic is so called because of its more-than-passing resemblance to Sir William Hamilton’s doctrine of the *quantification of the predicate*. (That is: Sir William Hamilton, Bart., the Scottish philosopher, not Sir William Hamilton, Kt., the Irish mathematician who discovered quaternions.) According to that doctrine, the predicates of the traditional syllogistic forms

$$\begin{array}{ll} \text{All } p \text{ are } q & \text{Some } p \text{ are } q \\ \text{No } p \text{ are } q & \text{Some } p \text{ are not } q, \end{array}$$

contain a suppressed existential quantifier (present, as Hamilton put it, *in thought*), which can meaningfully be dualized to yield the forms

$$\begin{array}{ll} \text{All } p \text{ are all } q & \text{Some } p \text{ are all } q \\ \text{No } p \text{ are all } q & \text{Some } p \text{ are not all } q. \end{array}$$

These latter sentences are of course, grammatically marginal, and we are owed an account of their purported meaning. Unfortunately, Hamilton’s presentation is hopelessly obscure in this regard: the closest we get to a formal account are the collections of diagrams in [3], pp. 682–683 and [4], p. 277. However, it is certain, from the accompanying text, that Hamilton took *All p are all q* to assert that *p* and *q* are coextensive—different from the meaning of the formula $\forall x(p(x) \rightarrow \forall y(q(y) \rightarrow x = y))$, and logically uninteresting.

Hamilton originally expounded his theory in an 1846 edition of the works of Sir Thomas Reid, in the form of a Prospectus of “An Essay towards a New Analytic of Logical Forms”, reproduced, with some variations, in the two sources mentioned above. The essay itself was never written. Nevertheless, Hamilton’s theory generated a heated debate with Augustus De Morgan, and was the subject of a critical commentary by John Stuart Mill ([7], Ch. XXII). The present paper attempts neither to resurrect that debate, nor to adjudicate its outcome. Unlike Hamilton’s quantified predicates, the sentence-forms in (2) and (3) are clearly grammatical, and the question of the existence of sound and complete proof procedures for this language consequently well-formed. Of course, to be

well-formed is one thing; to be well-motivated, another. We mention just one striking historical fact by way of justification. Notwithstanding their dubious grammatical status, copula sentences with quantified predicates make regular appearances in discussions of the syllogism, beginning with Aristotle himself (see [1], A27, 43a12–43b22). Indeed, Hamilton’s own extensive survey of this literature can be found in [4], pp. 298–317. Why, if we happily judge *No pacifist admires every quaker* to be grammatical, are we much less comfortable with *No pacifist is every quaker*? What, if anything, is this non-grammaticality judgement preventing us from expressing? And if there is something, what would be the logical consequences of expressing it anyway? Thus, our investigation of the Hamiltonian syllogistic addresses a venerable, if currently quiescent, issue.

On the basis of the equivalence of the forms (1) and (2), we take the best candidates for syllogistic forms with universally quantified predicates to be (3), interpreted as (4). We show in the sequel that, under this interpretation, no finite set of syllogistic rules can be sound and complete for the Hamiltonian syllogistic, a fact which distinguishes it from its classical subset. However, we do provide a finite set of such rules which is sound and *refutation*-complete—i.e. becomes complete if the rule of *reductio ad absurdum* is permitted as a final step. We go on to consider the effect of adding noun-level negation to the Hamiltonian syllogistic, yielding such sentence-forms as *No non-p is identical to every non-q*. We show that, unless $\text{PTIME} = \text{NPTIME}$, no finite set of syllogistic rules can be sound and refutation-complete for this extended language. However, we do provide a finite set of such rules which are sound and complete if the rule of *reductio ad absurdum* may be used without restriction. Such sensitivity to noun-level negation again distinguishes the Hamiltonian syllogistic from its classical subset.

By replacing the words *is identical to* in (2) and (3) by a suitably inflected transitive verb v (*admire*, *despise* ...), we obtain the forms

Every p <i>vs</i> some q	Some p <i>vs</i> some q
No p <i>vs</i> any q	Some p does not v any q
Every p <i>vs</i> every q	Some p <i>vs</i> every q
No p <i>vs</i> every q	Some p does not v every q .

This language was analysed in Pratt-Hartmann and Moss [8], where it was called the *relational syllogistic*. It was shown there that no finite set of syllogistic rules in the relational syllogistic is sound and complete, though there is a finite set of rules that is sound and refutation-complete. It was shown in the same paper that, when the relational syllogistic is extended with noun-level negation, there is no finite set of syllogistic rules that is sound and complete for the resulting language, even when the rule of *reductio ad absurdum* may be used without restriction. Thus, the Hamiltonian syllogistic differs in its proof-theoretic properties from the relational syllogistic as well.

2 Syntax and semantics

In this section, we define five formal languages: (i) \mathcal{S} , a formalization of the classical syllogistic, (ii) \mathcal{S}^\dagger , its extension with noun-level negation, (iii) \mathcal{H} , a formalization of the Hamiltonian syllogistic, (iv) \mathcal{H}^\dagger , its extension with noun-level negation, and (v) $\mathcal{H}^{*\dagger}$, an extension of \mathcal{H}^\dagger motivated chiefly by the formalism used below.

Fix a countably infinite set \mathbf{P} . We refer to any element of \mathbf{P} as an *atom*. A *literal* is an expression of either of the forms p or \bar{p} , where p is an atom. A literal which is an atom is called *positive*, otherwise, *negative*. If $\ell = \bar{p}$ is a negative literal, then we denote by $\bar{\ell}$ the positive literal p . A *structure* is a pair $\mathfrak{A} = \langle A, \{p^\mathfrak{A}\}_{p \in \mathbf{P}} \rangle$, where A is a non-empty set, and $p^\mathfrak{A} \subseteq A$, for every $p \in \mathbf{P}$. The set A is called the *domain* of \mathfrak{A} . We extend the map $p \mapsto p^\mathfrak{A}$ to negative literals by setting, for any atom p ,

$$\bar{p}^\mathfrak{A} = A \setminus p^\mathfrak{A}.$$

Intuitively, we may think of the elements of \mathbf{P} as common count-nouns, such as *pacifist*, *quaker*, *republican*, etc., and we think of $p^\mathfrak{A}$ as the set of things falling under the noun p according to the structure \mathfrak{A} . Thus, we may read \bar{p} as either *non- p* or *not a p* , depending on grammatical context.

An *\mathcal{S} -formula* is any expression of the forms

$$\begin{array}{lll} \forall(p, q) & \forall(p, \bar{q}) & \forall(\bar{p}, \bar{q}) \\ \exists(p, q) & \exists(p, \bar{q}) & \exists(\bar{p}, \bar{q}), \end{array} \quad (5)$$

where p and q are atoms; and an *\mathcal{S}^\dagger -formula* is any expression of the forms

$$\forall(\ell, m) \quad \exists(\ell, m), \quad (6)$$

where ℓ and m are literals. Thus, every \mathcal{S} -formula is an \mathcal{S}^\dagger -formula. If \mathfrak{A} is a structure, we write $\mathfrak{A} \models \forall(\ell, m)$ if $\ell^\mathfrak{A} \subseteq m^\mathfrak{A}$, and $\mathfrak{A} \models \exists(\ell, m)$ if $\ell^\mathfrak{A} \cap m^\mathfrak{A} \neq \emptyset$. We think of $\mathfrak{A} \models \varphi$ as asserting that φ is true in the structure \mathfrak{A} . Thus, we may read $\forall(\ell, m)$ as *Every ℓ is an m* and $\exists(\ell, m)$ as *Some ℓ is an m* . Under these semantics, the formulas $\exists(\ell, m)$ and $\exists(m, \ell)$ are true in exactly the same structures; and similarly for the formulas $\forall(\ell, m)$ and $\forall(\bar{m}, \bar{\ell})$. In the sequel, we identify these pairs of formulas, silently converting one into the other where needed. Taking account of these identifications, \mathcal{S} contains four different formulas (not six), which may be glossed by the sentence-forms (1) or, equivalently, (2). Likewise, \mathcal{S}^\dagger contains six different formulas (not eight), where $\forall(\bar{p}, q)$ is to be glossed as *Every non- p is a q* , and $\exists(\bar{p}, \bar{q})$ as *Some non- p is not a q* .

Turning now to the languages \mathcal{H} , \mathcal{H}^\dagger and $\mathcal{H}^{*\dagger}$, a *c-term* is either a literal or any expression of the forms $\forall p$ or $\forall \bar{p}$, where p is an atom; and an *e-term* is either a literal or any expression of the forms $\forall \ell$ or $\forall \bar{\ell}$, where ℓ is a literal. Thus, all literals are c-terms, and all c-terms are e-terms. If e is an e-term of the form $\forall \bar{\ell}$, we take \bar{e} to be the corresponding e-term $\forall \ell$. It follows that \bar{e} is a c-term if and only if e is, and \bar{e} is a literal if and only if e is; moreover, for any e-term e ,

$\bar{e} = e$. If \mathfrak{A} is a structure, we extend the map $\ell \mapsto \ell^{\mathfrak{A}}$ to non-literal e-terms by setting, for any literal ℓ ,

$$\begin{aligned} (\forall \ell)^{\mathfrak{A}} &= \{a \in A \mid a = b \text{ for all } b \in \ell^{\mathfrak{A}}\} \\ (\overline{\forall \ell})^{\mathfrak{A}} &= \{a \in A \mid a \neq b \text{ for some } b \in \ell^{\mathfrak{A}}\}, \end{aligned}$$

Thus, we may read $\forall \ell$ as *thing that is identical to every ℓ* , and $\overline{\forall \ell}$ as *thing that is not identical to every ℓ* (that is: *thing that is distinct from some ℓ*). Because terms of the form $\forall \ell$ can be confusing to parse in certain contexts, we sometimes enclose them in parentheses, thus: $(\forall \ell)$.

An \mathcal{H} -formula is any expression of the forms

$$\begin{array}{ll} \forall(p, c) & \forall(c, \bar{p}) \\ \exists(p, c) & \exists(c, p), \end{array}$$

where p is an atom and c is a c-term; an \mathcal{H}^{\dagger} -formula is any expression of the forms

$$\forall(\ell, e) \qquad \exists(\ell, e),$$

where ℓ is a literal and e is an e-term; and an $\mathcal{H}^{*\dagger}$ -formula is any expression of the forms

$$\forall(e, f) \qquad \exists(e, f),$$

where e and f are e-terms. Thus, every \mathcal{S}^{\dagger} -formula is an \mathcal{H} formula, every \mathcal{H} -formula is an \mathcal{H}^{\dagger} formula, and every \mathcal{H}^{\dagger} -formula is an $\mathcal{H}^{*\dagger}$ formula. We define $\mathfrak{A} \models \varphi$ for $\mathcal{H}^{*\dagger}$ -formulas in the same way as for \mathcal{S}^{\dagger} -formulas, again silently converting $\exists(e, f)$ to $\exists(f, e)$, and $\forall(e, f)$ to $\forall(\bar{f}, \bar{e})$, as needed. Taking account of these conversions, the eight forms of \mathcal{H} -formula may be glossed using the sentence-forms (2) and (3); and \mathcal{H}^{\dagger} -formulas may be similarly glossed, but using negated noun-phrases such as *non-pacifist*, *non-quaker*, etc. in the obvious way. Formulas of the language $\mathcal{H}^{*\dagger}$, by contrast, require more elaborate English translations: for example, $\forall(\forall \bar{p}, \overline{\forall q})$ may be glossed as:

Everything that is identical to every non- p is distinct from some q .

The primary motivation for considering the system $\mathcal{H}^{*\dagger}$ is the greater simplicity of its associated deduction system.

We denote the set of all \mathcal{S} -formulas by \mathcal{S} , and similarly for the other languages considered here. Where the language is clear from context, we speak simply of *formulas*. If $\varphi = \forall(e, f)$, we write $\bar{\varphi}$ to denote $\exists(e, \bar{f})$; and if $\varphi = \exists(e, f)$, we write $\bar{\varphi}$ to denote $\forall(e, \bar{f})$. Thus, $\bar{\bar{\varphi}} = \varphi$, and, in any structure \mathfrak{A} , $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A} \not\models \bar{\varphi}$. It is simple to check that, if \mathcal{L} is any of the languages \mathcal{S} , \mathcal{S}^{\dagger} , \mathcal{H} or \mathcal{H}^{\dagger} , then $\varphi \in \mathcal{L}$ implies $\bar{\varphi} \in \mathcal{L}$: that is, all the languages introduced above are, in effect, closed under negation. If Θ is a set of formulas, we write $\mathfrak{A} \models \Theta$ if, for all $\theta \in \Theta$, $\mathfrak{A} \models \theta$. A formula θ is *satisfiable* if there exists a structure \mathfrak{A} such that $\mathfrak{A} \models \theta$; a set of formulas Θ is *satisfiable* if there exists \mathfrak{A} such that $\mathfrak{A} \models \Theta$. If, for all structures \mathfrak{A} , $\mathfrak{A} \models \Theta$ implies $\mathfrak{A} \models \theta$, we say that Θ *entails* θ ,

and write $\Theta \models \theta$. We call a formula of the form $\exists(e, \bar{e})$ an *absurdity*, and use \perp to denote, indifferently, any absurdity. Evidently, \perp is unsatisfiable.

We illustrate the logics \mathcal{H} and \mathcal{H}^\dagger with some sample entailments. In the former case, we have, for example:

$$\{\exists(p, \forall q), \exists(q, o)\} \models \forall(q, o). \quad (7)$$

For suppose that some p is identical to every q , and there is a q which is also an o . Then there is exactly one q , and it is o ; therefore, every q is an o . In the latter case, we have, for example:

$$\{\forall(p, \forall p), \forall(\bar{p}, p), \exists(q_1, q_1)\} \models \forall(q_2, q_1) \quad (8)$$

$$\{\forall(p, \forall p), \forall(\bar{p}, \forall \bar{p}), \exists(q_1, \bar{q}_2), \exists(q_2, \bar{q}_3)\} \models \forall(q_3, q_1). \quad (9)$$

The validity (8) follows from the fact that any model of the premises has a 1-element domain. Likewise, in (9), any model of the premises has a 2-element domain. Thus, in the language \mathcal{H}^\dagger , it is possible to write satisfiable sets of formulas whose only models are of size 1 or 2. (This is trivially impossible in \mathcal{H} .) On the other hand, we shall see in Theorem 7.11 that, if a set of $\mathcal{H}^{*\dagger}$ -formulas has a model with three or more elements, then it has arbitrarily large models.

To ease readability in proofs, we employ the following variable-naming conventions. The variables o, p and q (possibly with decorations) are to be understood as ranging only over atoms, the variables ℓ and m only over literals, the variables c and d only over c-terms, and the variables e, f and g over e-terms. Thus, for example, if S is a set of e-terms, the statement “there exists $\ell \in S \dots$ ” should be read as “there exists a literal $\ell \in S \dots$ ”, and so on.

3 Proof Theory

By a *sylogistic language*, we mean any of the languages $\mathcal{S}, \mathcal{S}^\dagger, \mathcal{H}, \mathcal{H}^\dagger$ or $\mathcal{H}^{*\dagger}$. This enumerative definition could be replaced by a more general characterization; however, the details are not relevant to the concerns of this paper, and we avoid them. The problem of finding sound and complete rule-systems for the language \mathcal{S} was solved (independently) in Smiley [9], Corcoran [2] and Martin [5]. This result is strengthened marginally in Pratt-Hartmann and Moss [8] (as explained below), and extended to the language \mathcal{S}^\dagger . Here, we seek a system of sylogistic rules which generate exactly the entailments in the languages $\mathcal{H}, \mathcal{H}^\dagger$ and $\mathcal{H}^{*\dagger}$. Because our results will be partly negative in character, we adopt a relatively formal approach.

Let \mathcal{L} be a sylogistic language. A *sylogistic rule* in \mathcal{L} is a pair Θ/θ , where Θ is a finite set (possibly empty) of \mathcal{L} -formulas, and θ an \mathcal{L} -formula. We call Θ the *antecedents* of the rule, and θ its *consequent*. We generally display rules in ‘natural-deduction’ style. For example,

$$\frac{\exists(p, q) \quad \forall(q, o)}{\exists(p, o)} \quad , \quad \frac{\exists(p, q) \quad \forall(q, \bar{o})}{\exists(p, \bar{o})} \quad , \quad (10)$$

where p , q and o are atoms, are syllogistic rules in \mathcal{S} , (hence in any larger syllogistic language) corresponding to the traditional syllogisms *Darii* and *Ferio*, respectively. A rule is *valid* if its antecedents entail its consequent. Thus, the rules (10) are valid. As a further example, the following obvious generalizations of (10) are valid syllogistic rules in \mathcal{H} :

$$\frac{\exists(p, q) \quad \forall(q, \forall o)}{\exists(p, \forall o)} \quad \frac{\exists(p, q) \quad \forall(q, \overline{\forall o})}{\exists(p, \overline{\forall o})} . \quad (11)$$

Let \mathcal{L} be a syllogistic language and \mathbf{X} a set of syllogistic rules in \mathcal{L} ; and denote by $\mathbb{P}(\mathcal{L})$ the set of subsets of \mathcal{L} . A *substitution* is a function $g : \mathbf{P} \rightarrow \mathbf{P}$; we extend g to \mathcal{L} -formulas and to sets of \mathcal{L} -formulas in the obvious way. An *instance* of a syllogistic rule Θ/θ is the syllogistic rule $g(\Theta)/g(\theta)$, where g is a substitution. We define the *direct syllogistic derivation relation* $\vdash_{\mathbf{X}}$ to be the smallest relation on $\mathbb{P}(\mathcal{L}) \times \mathcal{L}$ satisfying:

1. if $\theta \in \Theta$, then $\Theta \vdash_{\mathbf{X}} \theta$;
2. if $\{\theta_1, \dots, \theta_n\}/\theta$ is a syllogistic rule in \mathbf{X} , g a substitution, $\Theta = \Theta_1 \cup \dots \cup \Theta_n$, and $\Theta_i \vdash_{\mathbf{X}} g(\theta_i)$ for all i ($1 \leq i \leq n$), then $\Theta \vdash_{\mathbf{X}} g(\theta)$.

Where the language \mathcal{L} is clear from context, we omit reference to it; further, we typically contract *syllogistic rule* to *rule*. Instances of the relation $\vdash_{\mathbf{X}}$ can always be established by *derivations* in the form of finite trees in the usual way. For instance, the derivation

$$\frac{\frac{\exists(p, q) \quad \forall(q, o)}{\exists(p, o)} \text{ (D1)} \quad \forall(o, \bar{r})}{\exists(p, \bar{r})} \text{ (D1)}$$

establishes that, for any set of syllogistic rules \mathbf{X} containing the rules (10),

$$\{\exists(p, q), \forall(q, o), \forall(o, \bar{r})\} \vdash_{\mathbf{X}} \exists(p, \bar{r}).$$

In the sequel, we reason freely about derivations in order to establish properties of derivation relations. The tags (D1) merely serve to indicate the rule employed in each step of the derivation: both the rules in (10) fall under a group which we shall later call (D1).

The syllogistic derivation relation $\vdash_{\mathbf{X}}$ is said to be *sound* if $\Theta \vdash_{\mathbf{X}} \theta$ implies $\Theta \models \theta$, and *complete* (for \mathcal{L}) if $\Theta \models \theta$ implies $\Theta \vdash_{\mathbf{X}} \theta$. A set Θ of formulas is *inconsistent* (with respect to $\vdash_{\mathbf{X}}$) if $\Theta \vdash_{\mathbf{X}} \perp$ for some absurdity \perp ; otherwise, *consistent*. It is obvious that, for any set of rules \mathbf{X} , $\vdash_{\mathbf{X}}$ is sound if and only if every rule in \mathbf{X} is valid. A weakening of completeness called *refutation-completeness* will prove important in the sequel: $\vdash_{\mathbf{X}}$ is *refutation-complete* if any unsatisfiable set Θ is inconsistent with respect to $\vdash_{\mathbf{X}}$. Completeness trivially implies refutation-completeness, but not conversely.

The languages \mathcal{H}^\dagger and $\mathcal{H}^{*\dagger}$ turn out to require a stronger form of proof-system than that provided by direct derivation relations. Let \mathcal{L} be a syllogistic language and \mathbf{X} a set of syllogistic rules in \mathcal{L} . We define the *indirect syllogistic derivation relation* $\Vdash_{\mathbf{X}}$ to be the smallest relation on $\mathbb{P}(\mathcal{L}) \times \mathcal{L}$ satisfying:

1. if $\theta \in \Theta$, then $\Theta \Vdash_X \theta$;
2. if $\{\theta_1, \dots, \theta_n\}/\theta$ is a syllogistic rule in X , g a substitution, $\Theta = \Theta_1 \cup \dots \cup \Theta_n$, and $\Theta_i \Vdash_X g(\theta_i)$ for all i ($1 \leq i \leq n$), then $\Theta \Vdash_X g(\theta)$.
3. if $\Theta \cup \{\theta\} \Vdash_X \perp$, where \perp is any absurdity, then $\Theta \Vdash_X \bar{\theta}$.

The only difference is the addition of the final clause, which allows us to derive a formula $\bar{\theta}$ from premises Θ if we can derive an absurdity from Θ together with θ . Instances of the indirect derivation relation \Vdash_X may also be established by constructing derivations, except that we need a little more machinery to keep track of premises. This may be done as follows. Suppose we have a derivation (direct or indirect) showing that $\Theta \cup \{\theta\} \Vdash_X \perp$, for some absurdity \perp . Let this derivation be displayed as

$$\begin{array}{ccccccc} \theta_1 & \dots & \theta_n & \theta & \dots & \theta & \\ & & & \vdots & & & \\ & & & \perp, & & & \end{array}$$

where $\theta_1, \dots, \theta_n$ is a list of formulas of Θ (not necessarily exhaustive, and with repeats allowed). Applying Clause 3 of the definition of \Vdash_X , we have $\Theta \Vdash_X \bar{\theta}$, which we take to be established by the derivation

$$\begin{array}{ccccccc} \theta_1 & \dots & \theta_n & [\theta]^1 & \dots & [\theta]^1 & \\ & & & \vdots & & & \\ & & & \perp & & & \\ \hline & & & \bar{\theta} & & & \end{array} \text{ (RAA)}^1.$$

The tag (RAA) stands for *reductio ad absurdum*; the square brackets indicate that the enclosed instances of θ have been *discharged*, i.e. no longer count among the premises; and the numerical indexing is simply to make the derivation history clear. Note that there is nothing to prevent θ from occurring among the $\theta_1, \dots, \theta_n$; that is to say, we do not have to discharge all (or indeed any) instances of the premise θ if we do not want to. Again, it should be obvious that, for any set of rules X , \Vdash_X is sound if and only if every rule in X is valid, and \Vdash_X is complete if it is refutation complete. It is important to understand that *reductio ad absurdum* cannot be formulated as a syllogistic rule in the technical sense defined here; rather, it is part of the proof-theoretic machinery that converts any set of rules X into the derivation relation \Vdash_X .

Syllogistic rules that differ only by renaming of atoms have the same sets of instances, and so may be regarded as identical. That is, in rules such as (10) and (11), we may informally think of the atoms o , p and q as metavariables ranging over the set of atoms \mathbf{P} . This suggests the following notational convention. Taking the meta-variable c to range over c-terms, we may comprehend the four rules in (10) and (11) under the single schema

$$\frac{\exists(p, q) \quad \forall(q, c)}{\exists(p, c)} \text{ (D1)}.$$

We shall employ this schematic notation in the sequel. Note, however, that such schemata are always shorthand for a *finite* number of rules. In the sequel, we generally refer to rule schemata simply as *rules*.

The following complexity-theoretic observations on derivation relations will prove useful in this paper.

Lemma 3.1. *Let \mathcal{L} be a syllogistic language, $\theta \in \mathcal{L}$ and $\Theta \subseteq \mathcal{L}$. If there is a derivation (direct or indirect) of θ from Θ using some set of rules X , then there is such a derivation involving only the atoms occurring in $\Theta \cup \{\theta\}$.*

Proof. Given a derivation of θ from Θ , uniformly replace any unary atom that does not occur in $\Theta \cup \{\theta\}$ with one that does. \square

Proposition 3.2. *Let \mathcal{L} be a syllogistic language, and X a finite set of syllogistic rules in \mathcal{L} . The problem of determining whether $\Theta \vdash_{\mathsf{X}} \theta$, for a given set of \mathcal{L} -formulas Θ and \mathcal{L} -formula θ , is in PTIME. Hence, if \vdash_{X} is sound and refutation-complete, the satisfiability problem for \mathcal{L} is in PTIME.*

Proof. By Lemma 3.1, we may confine attention to derivations featuring only the atoms in $\Theta \cup \{\theta\}$. The number m of \mathcal{L} -formulas featuring these atoms is bounded by a quadratic function of $|\Theta \cup \{\theta\}|$; evidently, we need only consider derivations with m or fewer steps. Suppose that the maximum number of premises of any rule in X is ℓ . If $k < m$, and the set of formulas derivable in k steps has been computed, then we may evidently compute the number of formulas derivable in $k + 1$ steps in time $O(m^{\ell+1})$. \square

Note that Proposition 3.2 does not apply to indirect derivation relations. However, we do have a weaker global complexity bound, even in this case. If Φ is set of \mathcal{L} -sentences, we say that Φ is *complete* if, for every \mathcal{L} -sentence φ featuring only the atoms occurring in Φ , either $\varphi \in \Phi$ or $\bar{\varphi} \in \Phi$. Trivially, every satisfiable set of \mathcal{L} -sentences can be extended to a complete, satisfiable set of \mathcal{L} -sentences. (Do not confuse this observation with Lemma 3.4.)

Proposition 3.3. *Let \mathcal{L} be a syllogistic language, X a finite set of syllogistic rules in \mathcal{L} , and Ψ a complete set of \mathcal{L} -sentences. If $\Psi \Vdash_{\mathsf{X}} \perp$, then $\Psi \vdash_{\mathsf{X}} \perp$. Hence, if \Vdash_{X} is sound and complete, the satisfiability problem for \mathcal{L} is in NPTIME.*

Proof. For the first statement, suppose that there is an indirect derivation of some absurdity \perp from Ψ , using the rules X . Let the number of applications of (RAA) employed in this derivation be k ; and assume without loss of generality that \perp is chosen so that this number k is minimal. If $k > 0$, consider the last application of (RAA) in this derivation, which derives a formula, say, $\bar{\psi}$, discharging a premise ψ . Then there is an (indirect) derivation of some absurdity \perp' from $\Psi \cup \{\psi\}$, employing fewer than k applications of (RAA). By minimality of k , $\psi \notin \Psi$, and so, by the completeness of Ψ , $\bar{\psi} \in \Psi$. But then we can replace our original derivation of $\bar{\psi}$ with the trivial derivation, so obtaining a derivation of \perp from Ψ with fewer than k applications of (RAA), a contradiction.

Therefore, $k = 0$, or, in other words, $\Psi \vdash_X \perp$. For the second statement, let a set of \mathcal{L} -sentences Φ be given. Now guess a complete superset Ψ involving only those atoms occurring in Φ . Evidently, $|\Psi|$ is bounded by a polynomial function of $|\Phi|$. By Proposition 3.2, we can check in polynomial time whether $\Psi \vdash_X \perp$. \square

We mentioned above that the existence of sound and complete syllogistic systems for the languages \mathcal{S} and \mathcal{S}^\dagger has been solved. More specifically, it is shown in Pratt-Hartmann and Moss [8], that, for both languages, a finite set of rules exist for which the associated direct derivation relation is sound and complete. (The earlier work cited above showed only the existence of sound and refutation-complete systems for \mathcal{S} .) We are now in a position to state the technical results of this paper:

1. There is no finite set X of syllogistic rules in \mathcal{H} such that \vdash_X is sound and complete (Theorem 4.1).
2. There is a finite set H of syllogistic rules in \mathcal{H} such that \vdash_H is sound and refutation-complete (Theorem 5.1).
3. The problem of determining whether a set of \mathcal{H}^\dagger -formulas is satisfiable is NPTIME-complete, and similarly for the problem of determining whether a set of $\mathcal{H}^{*\dagger}$ -formulas is satisfiable (Theorem 6.1). Hence, by Proposition 3.2, unless PTIME=NPTIME, there is no finite set X of syllogistic rules in either \mathcal{H}^\dagger or $\mathcal{H}^{*\dagger}$ such that \vdash_X is sound and refutation-complete.
4. There is a finite set H^\dagger of syllogistic rules in \mathcal{H}^\dagger such that \Vdash_{H^\dagger} is sound and complete (Theorem 7.1).
5. There is a finite set $H^{*\dagger}$ of syllogistic rules in $\mathcal{H}^{*\dagger}$ such that $\Vdash_{H^{*\dagger}}$ is sound and complete (Theorem 7.2).

The following sections of this paper are devoted to proofs of these results. We round off the present section by establishing a version of the Lindenbaum Lemma for indirect derivation relations. This result will be used in Section 7.

Lemma 3.4. *Let \mathcal{L} be a syllogistic language, X a finite set of syllogistic rules in \mathcal{L} , and Φ a set of \mathcal{L} -formulas. If Φ is \Vdash_X -consistent, then Φ has a \Vdash_X -consistent, complete extension.*

Proof. Enumerate the \mathcal{L} -formulas as $\varphi_0, \varphi_1, \dots$. Define $\Phi_0 = \Phi$, and

$$\Phi_{i+1} = \begin{cases} \Phi \cup \{\varphi_i\} & \text{if } \Phi \not\Vdash_X \bar{\varphi}_i \\ \Phi \cup \{\bar{\varphi}_i\} & \text{otherwise,} \end{cases}$$

for all $i \geq 0$. We show by induction that each Φ_i is consistent. From this it follows that $\Phi^* = \bigcup_{0 \leq i} \Phi_i$ is consistent, thus proving the lemma. The case $i = 0$ is true by hypothesis; so we suppose that Φ_i is consistent, but Φ_{i+1} inconsistent, and derive a contradiction. Assume first that $\Phi_i \not\Vdash_X \bar{\varphi}_i$. Thus,

$\Phi_{i+1} = \Phi \cup \{\varphi_i\} \Vdash_X \perp$, whence, by the rule (RAA), $\Phi_i \Vdash_X \bar{\varphi}_i$, contrary to assumption. On the other hand, assume $\Phi_i \Vdash_X \bar{\varphi}_i$, so that $\Phi_{i+1} = \Phi_i \cup \{\bar{\varphi}_i\}$. Take derivations establishing that $\Phi_i \Vdash_X \bar{\varphi}_i$ and that $\Phi_i \cup \{\bar{\varphi}_i\} \Vdash_X \perp$; and chain these together to form a single derivation, thus:

$$\begin{array}{c} \Phi_i \\ \vdots \\ \Phi_i, \bar{\varphi}_i \\ \vdots \\ \perp \end{array} .$$

This establishes that $\Phi_i \Vdash_X \perp$, contrary to the supposed consistency of Φ . \square

4 No complete syllogistic systems for \mathcal{H}

The objective of this section is to prove

Theorem 4.1. *There is no finite set X of syllogistic rules in \mathcal{H} such that \vdash_X is sound and complete.*

We use a variant of a technique from Pratt-Hartmann and Moss [8]. For $n \geq 3$, let Γ^n be the set of formulas

$$\forall(p_i, \overline{\forall p_{i+1}}) \quad (1 \leq i < n) \quad (12)$$

$$\forall(p_1, \forall p_n) \quad (13)$$

$$\forall(p_n, \forall p_1) \quad (14)$$

$$\forall(p_i, p_i) \quad (1 \leq i \leq n) \quad (15)$$

$$\forall(p_1, \bar{p}_{n-1}) \quad (16)$$

and let γ^n be the formula $\forall(p_1, p_n)$. Note that the Formulas (13) and (14) are logically equivalent, that Formulas (15) are true in every structure, and that Formula (16) is an immediate consequence of (12) (putting $i = n - 1$) and (13). Further, $\Gamma^n \models \gamma^n$. To see this, suppose for contradiction that $\mathfrak{A} \models \Gamma^n$, but $a \in p_1^{\mathfrak{A}} \setminus p_n^{\mathfrak{A}}$. Since $p_1^{\mathfrak{A}} \neq \emptyset$, the formulas (12) ensure that $p_i^{\mathfrak{A}} \neq \emptyset$ for all i ($1 \leq i \leq n$). By (13), then, a is the unique element of $p_n^{\mathfrak{A}}$. But this contradicts the fact that $a \notin p_n^{\mathfrak{A}}$. We proceed to show that, for any finite set X of syllogistic rules, if \vdash_X is sound, then there exists a value of n such that $\Gamma^n \not\vdash_X \gamma^n$.

For any h , $1 \leq h \leq n - 2$, define $\Gamma_h^n = \Gamma^n \setminus \{\forall(p_h, \overline{\forall p_{h+1}})\}$.

Lemma 4.2. *Let φ be an \mathcal{H} -formula featuring only the atoms p_1, \dots, p_n , and let $1 \leq h \leq n - 2$. Then either $\Gamma_h^n \not\models \varphi$ or $\varphi \in \Gamma^n$.*

Proof. We consider the possible forms of φ in turn.

1. $\varphi = \forall(p_i, p_j)$: Let $A = \{a_1, \dots, a_{n-1}\}$, and define the structure \mathfrak{A} over A by setting

$$p_k^{\mathfrak{A}} = \{a_k\} \quad (1 \leq k < n), \quad p_n^{\mathfrak{A}} = \{a_1\}.$$

A routine check shows that $\mathfrak{A} \models \Gamma_h^n$, but, for $i \neq j$ and $\{i, j\} \neq \{1, n\}$, $\mathfrak{A} \not\models \varphi$. On the other hand, if $i = j$ then, from (15), $\varphi \in \Gamma^n$. This means we need only deal with the case $\{i, j\} = \{1, n\}$. For all h ($1 \leq h \leq n-2$), let $C_h = \{a_1, \dots, a_h\}$ and $D_h = \{a_{h+1}, \dots, a_n\}$, and define the structures \mathfrak{C}_h and \mathfrak{D}_h by setting:

$$\begin{aligned} p_k^{\mathfrak{C}_h} &= \{a_k\} & (1 \leq k \leq h) & & p_k^{\mathfrak{C}_h} &= \emptyset & (h < k \leq n) \\ p_k^{\mathfrak{D}_h} &= \emptyset & (1 \leq k \leq h) & & p_k^{\mathfrak{D}_h} &= \{a_k\} & (h < k \leq n). \end{aligned}$$

A routine check shows that $\mathfrak{C}_h \models \Gamma_h^n$ and $\mathfrak{D}_h \models \Gamma_h^n$, but that $\mathfrak{C}_h \not\models \forall(p_1, p_n)$ and $\mathfrak{D}_h \not\models \forall(p_n, p_1)$.

2. $\varphi = \exists(p_i, c)$: It is immediate that

$$\begin{aligned} \mathfrak{C}_h &\not\models \exists(p_i, c) & (h < i < n \text{ and } c \text{ any c-term}) \\ \mathfrak{D}_h &\not\models \exists(p_i, c) & (1 \leq i \leq h \text{ and } c \text{ any c-term}) \end{aligned}$$

Hence, if $i \leq h$, $\mathfrak{D}_h \models \Gamma_h^n$, but $\mathfrak{D}_h \not\models \varphi$; if $i > h$, $\mathfrak{C}_h \models \Gamma_h^n$, but $\mathfrak{C}_h \not\models \varphi$.

3. $\varphi = \forall(p_i, \bar{p}_j)$: Given that the formulas $\forall(p_i, \bar{p}_j)$ and $\forall(p_j, \bar{p}_i)$ are identified in this paper, we may assume without loss of generality that $i \leq j$. If $i = j$, or if $i = 1$ and $j = n$, then $\mathfrak{A} \models \varphi$. If $i = 1$ and $j = n-1$, then, from (16), we have $\varphi \in \Gamma^n$. We next suppose that either $1 \leq i < j \leq n-2$, or $2 \leq i < j \leq n-1$. For such values of i and j , define $\mathfrak{A}_{i,j}$ over A by setting

$$p_k^{\mathfrak{A}} = \{a_k\} \quad (1 \leq k < n) \text{ and } k \neq i, \quad p_i^{\mathfrak{A}} = \{a_i, a_j\}, \quad p_n^{\mathfrak{A}} = \{a_1\}.$$

Thus, $\mathfrak{A}_{i,j}$ is just like \mathfrak{A} , except that the element a_i additionally realizes the predicate p_j . A routine check shows that $\mathfrak{A}_{i,j} \models \Gamma_h^n$, but $\mathfrak{A}_{i,j} \not\models \varphi$. (Note that $\mathfrak{A}_{i,j}$ is not defined if $j = n$ or if $i = 1$ and $j = n-1$.) We next suppose that $2 \leq i \leq n-2$ and $j = n$. Define \mathfrak{A}_i to be just like \mathfrak{A} , except that the first element a_1 additionally realizes the predicate p_i . A routine check shows that $\mathfrak{A}_i \models \Gamma_h^n$, but $\mathfrak{A}_i \not\models \varphi$. (Note that \mathfrak{A}_i is not defined if $i = 1$ or $n-1 \leq i \leq n$.) The only remaining case is where $i = n-1$ and $j = n$. Define the structure \mathfrak{D}'_h to be just like \mathfrak{D}_h , except that a_{n-1} additionally satisfies the predicate p_n . Again, a routine check shows that $\mathfrak{D}'_h \models \Gamma_h^n$, but $\mathfrak{D}'_h \not\models \varphi$.

4. $\varphi = \forall(p_i, \forall p_j)$: Given that Formulas (13) and (14) are the only formulas of this form in Γ^n , we may assume without loss of generality that $i \leq j$, and also that either $1 < i$ or $j < n$. If $1 \leq i < j \leq n$ and $\{i, j\} \neq \{1, n\}$, then $\mathfrak{A} \models \varphi$. This leaves only the case where $i = j$. Denote by $2 \times \mathfrak{C}_h$ the resulting of taking two disjoint copies of \mathfrak{C}_h , and similarly for $2 \times \mathfrak{D}_h$. A routine check shows that $2 \times \mathfrak{C}_h \models \Gamma_h^n$ and $2 \times \mathfrak{D}_h \models \Gamma_h^n$. On the other hand

$$\begin{aligned} 2 \times \mathfrak{C}_h &\not\models \forall(p_i, \forall p_i) & 1 \leq i \leq h \\ 2 \times \mathfrak{D}_h &\not\models \forall(p_i, \forall p_i) & h < i < n. \end{aligned}$$

Thus, if $i = j \leq h$, then $2 \times \mathfrak{C}_h \not\models \varphi$, and if $i = j > h$, then $2 \times \mathfrak{D}_h \not\models \varphi$.

5. $\varphi = \forall(p_i, \overline{\forall p_j})$: If $i = j$, then $\mathfrak{A} \not\models \varphi$. If $j = i + 1$, then, from (12), $\varphi \in \Gamma^n$. If $1 \leq i \leq h < j \leq n$, then $\mathfrak{C}_h \not\models \varphi$. If $1 \leq j \leq h < i \leq n$, then $\mathfrak{D}_h \not\models \varphi$. If $1 \leq i \leq h$, $1 \leq j \leq h$ and $j \neq i + 1$, let the structure $\mathfrak{C}_{h,i,j}$ be just like \mathfrak{C}_h , except that a_j additionally satisfies the predicate p_i . A routine check shows that $\mathfrak{C}_{h,i,j} \models \Gamma_h^n$, but $\mathfrak{C}_{h,i,j} \not\models \varphi$. (Note that $\mathfrak{C}_{h,i,j}$ is not defined if $j = i + 1$.) Similarly, if $h < i \leq n$, $h < j \leq n$ and $j \neq i + 1$, let the structure $\mathfrak{D}_{h,i,j}$ be just like \mathfrak{D}_h , except that a_j additionally satisfies the predicate p_i . Again, we have $\mathfrak{D}_{h,i,j} \models \Gamma_h^n$, but $\mathfrak{D}_{h,i,j} \not\models \varphi$. \square

Proof of Theorem 4.1. Let \mathbf{X} be a finite (non-empty) set of syllogistic rules such that $\vdash_{\mathbf{X}}$ is sound. Let the maximum number of antecedents in any of the rules of \mathbf{X} be $r \geq 0$, fix $n = r + 3$, and let θ be any \mathcal{H} -formula featuring only the atoms p_1, \dots, p_n . We claim that $\Gamma^{(n)} \vdash_{\mathbf{X}} \theta$ implies $\theta \in \Gamma^n$. Since $\gamma^n \notin \Gamma^n$, this proves the theorem.

We prove the claim by induction on the lengths of derivations. By Lemma 3.1, if there is a derivation of γ^n from Γ^n , then there is such a derivation using only the atoms p_1, \dots, p_n . Henceforth, then, we confine ourselves to derivations featuring only these atoms. Now, for derivations employing no steps of inference—i.e. for $\theta \in \Gamma^n$ —the claim is trivial. So suppose that the claim holds for derivations employing at most p steps, and that θ is derived from Γ in $p + 1$ steps. By inductive hypothesis, the antecedents of the final rule-instance will all be in Γ^n ; therefore, since $n = r + 3$, the antecedents of the final rule-instance will all be in Γ_h^n , for some h ($1 \leq h \leq n - 2$). Since $\vdash_{\mathbf{X}}$ is sound, $\Gamma_h^n \vdash \theta$, whence, by Lemma 4.2, $\theta \in \Gamma^n$. This completes the inductive step, and the proof of the theorem. \square

5 A refutation-complete syllogistic system for \mathcal{H}

The objective of this section is to prove

Theorem 5.1. *There is a finite set \mathbf{H} of syllogistic rules in \mathcal{H} such that the direct derivation relation $\vdash_{\mathbf{H}}$ is sound and refutation-complete.*

We display \mathbf{H} in schematic form, with o , p and q ranging over atoms, and c over c -terms, as usual. The rule-schemata fall naturally into four groups.

1. ‘little’ rules:

$$\frac{\exists(p, c)}{\exists(p, p)} \text{ (I)} \quad \frac{}{\forall(p, p)} \text{ (T)};$$

2. rules similar to familiar syllogisms:

$$\frac{\forall(p, q) \quad \forall(q, c)}{\forall(p, c)} \text{ (B)}$$

$$\frac{\exists(p, q) \quad \forall(q, c)}{\exists(p, c)} \text{ (D1)} \quad \frac{\exists(p, c) \quad \forall(p, q)}{\exists(q, c)} \text{ (D2)} \quad \frac{\exists(p, c) \quad \forall(q, \bar{c})}{\exists(p, \bar{q})} \text{ (D3)};$$

3. ‘little’ rules for universally quantified predicates:

$$\frac{\exists(p, \forall q)}{\forall(q, p)} \text{ (H1)} \quad \frac{\exists(p, \forall q)}{\forall(q, \forall q)} \text{ (H2)} \quad \frac{\exists(p, \overline{\forall q})}{\exists(q, \overline{\forall p})} \text{ (H3)};$$

4. syllogism-like rules for universally quantified predicates

$$\begin{array}{ll} \frac{\exists(q, c) \quad \exists(p, \forall q)}{\forall(q, c)} \text{ (HH1)} & \frac{\exists(p, c) \quad \forall(p, \forall q)}{\forall(q, c)} \text{ (HH2)} \\ \frac{\forall(p, c) \quad \exists(p, \forall q)}{\forall(q, c)} \text{ (HH3)} & \frac{\forall(p, \forall q) \quad \exists(q, q)}{\forall(p, q)} \text{ (HH4)}. \end{array}$$

Note that these rule-schemata define a finite set of rules, as explained above. Our choice of labels (I), (T), etc. is essentially arbitrary, though (B), (D1), (D2) and (D3) allude vaguely to the classical syllogisms *Barbara* and *Darii*. Recalling our decision silently to identify the formulas $\forall(p, c)$ and $\forall(\bar{c}, \bar{p})$, (D3) could be alternatively written as $\{\exists(p, c), \forall(c, \bar{q})\} / \exists(p, \bar{q})$. Validity of these rules is transparent: Rule (HH1) is a straightforward generalization of the validity (7) considered above; the other ‘Hamiltonian’ rules are dealt with similarly.

Let Φ be a set of \mathcal{H} -formulas containing at least one existential formula, such that Φ is consistent with respect to $\vdash_{\mathbf{H}}$. In the following lemmas, we build a structure \mathfrak{A} , and show that $\mathfrak{A} \models \Phi$. Since \mathbf{H} is the only set of rules we shall be concerned with in this section, we write \vdash for the direct proof-relation $\vdash_{\mathbf{H}}$. We remind the reader that the variables o , p and q are silently assumed to range only over atoms, and the variables c and d over c-terms.

Let S be a set of c-terms. We define S^* to be the smallest set of c-terms including S such that, for all atoms p , q and all c-terms c :

$$p \in S^* \text{ and } \Phi \vdash \forall(p, c) \Rightarrow c \in S^* \quad (\text{C1})$$

$$(\forall p) \in S^* \text{ and } \Phi \vdash \exists(p, p) \Rightarrow p \in S^*. \quad (\text{C2})$$

Evidently, we may regard S^* as the limit of a process in which, starting with S , c-terms are added one by one to ensure fulfillment of the above conditions. More precisely, we may write $S^* = \bigcup_{0 \leq i < \alpha} S^{(i)}$, where $S^{(0)} = S$, $\alpha \leq \omega$, and, for all i ($i+1 < \alpha$), $S^{(i+1)} = S^{(i)} \cup \{c\}$ for some $c \notin S^{(i)}$ satisfying either of the following conditions:

$$\text{there exists } p \in S^{(i)} \text{ such that } \Phi \vdash \forall(p, c); \quad (\text{K1})$$

$$c = p \text{ is an atom such that } (\forall p) \in S^{(i)}, \text{ and } \Phi \vdash \exists(p, p). \quad (\text{K2})$$

Define the set W to be the following set of c-terms:

$$\begin{array}{ll} W_0 &= \{\{p, c\}^* \mid \Phi \vdash \exists(p, c)\} \\ W_{i+1} &= \{\{p\}^* \mid \overline{\forall p} \in w \text{ for some } w \in W_i\} \quad (i \geq 0) \\ W &= \bigcup_{i \geq 0} W_i. \end{array}$$

Since Φ contains at least one existential formula, W is non-empty. We use letters u, v, w to range over elements of W . Lemmas 5.2–5.7 establish some properties of W .

Lemma 5.2. *Let $c \in w \in W$. Then there exists $o \in w$ such that $\Phi \vdash \exists(o, c)$.*

Proof. Assume first that $w \in W_0$. Thus, $w = \{o, d\}^*$, where $\Phi \vdash \exists(o, d)$. Using the representation $w = \bigcup_{0 \leq i < \alpha} S^{(i)}$, where $S^{(0)} = \{o, d\}$, we show by induction on i that, if $c \in S^{(i)} \in W$, there exists $o \in w$ such that $\Phi \vdash \exists(o, c)$.

For $i = 0$, we have $c = d$ or $c = o$. In the former case, $\Phi \vdash \exists(o, c)$, by assumption. In the latter, we have the derivation

$$\frac{\vdots}{\frac{\exists(o, d)}{\exists(o, o)}} \text{ (I),}$$

so that, either way, $\Phi \vdash \exists(o, c)$. For $i \geq 1$, we consider the following cases, corresponding to the conditions (K1)–(K2).

1. $\Phi \vdash \forall(q, c)$ for some $q \in S^{(i-1)}$: By inductive hypothesis, there exists $o \in w$ such that $\Phi \vdash \exists(o, q)$, so we have the derivation

$$\frac{\frac{\vdots}{\exists(o, q)} \quad \frac{\vdots}{\forall(q, c)}}{\exists(o, c)} \text{ (D1).}$$

2. $c = q, \forall q \in S^{(i-1)}$, and $\Phi \vdash \exists(q, q)$: But then there is nothing to show, since we may put $o = q$.

This completes the proof of the lemma for $w \in W_0$. We now prove the result for $w \in W_k$, for all $k \geq 0$, proceeding by induction on k . For $k > 0$, we have $w = \{o\}^*$, where, for some $v \in W_{k-1}$, $\overline{\forall} o \in v$. By inductive hypothesis, there exists $p \in v$ such that $\Phi \vdash \exists(p, \overline{\forall} o)$, so we have the derivation

$$\frac{\frac{\vdots}{\exists(p, \overline{\forall} o)} \text{ (H3)}}{\frac{\exists(o, \overline{\forall} p)}{\exists(o, o)} \text{ (I).}}$$

Having established that $\Phi \vdash \exists(o, o)$, we can proceed exactly as for the case $k = 0$, writing $w = \bigcup_{0 \leq i < \alpha} S^{(i)}$, where $S^{(0)} = \{o\}$. \square

Lemma 5.3. *Let $p \in w \in W$. Then $\Phi \vdash \exists(p, p)$.*

Proof. By Lemma 5.2, let o be such that $\Phi \vdash \exists(o, p)$. Then we have the derivation

$$\frac{\vdots}{\exists(o, p)} \quad \frac{\vdots}{\exists(p, p)} \quad (\text{I}).$$

□

Lemma 5.4. *If $c \in \{p\}^*$, then $\Phi \vdash \forall(p, c)$.*

Proof. Write $\{p\}^* = \bigcup_{0 \leq i < \alpha} S^{(i)}$, with $S^{(0)} = \{p\}$, as in the proof of Lemma 5.2; we show that the lemma holds for $c \in S^{(i)}$, proceeding by induction on i .

If $i = 0$, then $c = p$, so $\Phi \vdash \forall(p, c)$ by rule (T). If $i \geq 1$, we again have two cases corresponding to the conditions (K1) and (K2).

1. $\Phi \vdash \forall(q, c)$ for some $q \in S^{(i-1)}$: By inductive hypothesis, $\Phi \vdash \forall(p, q)$, so we have the derivation

$$\frac{\frac{\vdots}{\forall(p, q)} \quad \frac{\vdots}{\forall(q, c)}}{\forall(p, c)} \quad (\text{B}).$$

2. $c = q$, $\forall q \in S^{(i-1)}$, and $\Phi \vdash \exists(q, q)$: By inductive hypothesis, $\Phi \vdash \forall(p, \forall q)$, so we have the derivation

$$\frac{\frac{\vdots}{\forall(p, \forall q)} \quad \frac{\vdots}{\exists(q, q)}}{\forall(p, q)} \quad (\text{HH4}).$$

In both cases, $\Phi \vdash \forall(p, c)$, as required. □

In the next lemma, we take $\bar{\forall}$ to be the symbol \exists and $\bar{\exists}$ to be the symbol \forall .

Lemma 5.5. *Suppose $c, d \in w \in W$ with c, d distinct. Then there exist $o \in w$ and $Q \in \{\forall, \exists\}$ such that $\Phi \vdash Q(o, c)$ and $\Phi \vdash \bar{Q}(o, d)$. Hence, if $c \in w$, then $\bar{c} \notin w$.*

Proof. We consider first the case $w \in W \setminus W_0$. By construction of W , $w = \{o\}^*$ for some atom o . By Lemma 5.3, $\Phi \vdash \exists(o, o)$; and by Lemma 5.4, $\Phi \vdash \forall(o, c)$ and $\Phi \vdash \forall(o, d)$. But then we have the derivation:

$$\frac{\frac{\vdots}{\exists(o, o)} \quad \frac{\vdots}{\forall(o, d)}}{\exists(o, d)} \quad (\text{D1}).$$

Henceforth, then, we may suppose $w \in W_0$, and we again write $w = \bigcup_{0 \leq i < \alpha} S^{(i)}$, as in the proof of Lemma 5.2. Note that $S^{(0)} = \{o, c'\}$, for some atom o and

c-term c' such that $\Phi \vdash \exists(o, c')$. We prove the lemma for $c \in S^{(i)}$ and $d \in S^{(j)}$, proceeding by induction on $i+j$, showing in fact that the required o lies in $S^{(0)}$.

If $i+j = 0$ —i.e., $c, d \in S^{(0)}$ —then, since c, d are distinct, we have $\{c, d\} = \{o, c'\}$ and $w \in W_0$. The result then follows immediately from the fact that, by rule (T), $\Phi \vdash \forall(o, o)$. If $i+j > 0$, assume without loss of generality that $i > 0$. We again have two cases corresponding to the conditions (K1) and (K2).

1. $\Phi \vdash \forall(q, c)$ for some $q \in S^{(i-1)}$: By inductive hypothesis, there exist $o \in S^{(0)}$ and $Q \in \{\forall, \exists\}$ such that $\Phi \vdash Q(o, q)$ and $\Phi \vdash \overline{Q}(o, d)$. We then have one of the derivations:

$$\frac{\begin{array}{c} \vdots \\ \forall(o, q) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q, c) \end{array}}{\forall(o, c)} \text{ (B)} \qquad \frac{\begin{array}{c} \vdots \\ \exists(o, q) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q, c) \end{array}}{\exists(o, c)} \text{ (D1)}.$$

so that $\Phi \vdash Q(o, c)$, as required.

2. $c = q, \forall q \in S^{(i-1)}$ and $\Phi \vdash \exists(q, q)$: By inductive hypothesis, there exists $o \in S^{(0)}$ and $Q \in \{\forall, \exists\}$ such that $\Phi \vdash Q(o, \forall q)$ and $\Phi \vdash \overline{Q}(o, d)$. Then we have one of the derivations

$$\frac{\begin{array}{c} \vdots \\ \forall(o, \forall q) \end{array} \quad \begin{array}{c} \vdots \\ \exists(q, q) \end{array}}{\forall(o, q)} \text{ (HH4)} \qquad \frac{\begin{array}{c} \vdots \\ \exists(q, q) \end{array} \quad \frac{\begin{array}{c} \vdots \\ \exists(o, \forall q) \end{array} \quad \begin{array}{c} \vdots \\ \forall(q, o) \end{array}}{\forall(q, o)} \text{ (H1)}}{\exists(o, q)} \text{ (D1)}.$$

For the final statement of the lemma, suppose $c \in w$ and $\bar{c} \in w$. Exchanging c and \bar{c} if necessary, let o be such that $\Phi \vdash \exists(o, c)$ and $\Phi \vdash \forall(o, \bar{c})$. Then we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \exists(o, c) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, \bar{c}) \end{array}}{\exists(o, \bar{o})} \text{ (D3)},$$

contradicting the supposed consistency of Φ . □

Lemma 5.6. *Let $p, c \in w \in W$. Then $\Phi \vdash \exists(p, c)$.*

Proof. If $p = c$, we can apply Lemma 5.3. Otherwise, by Lemma 5.5, let $o \in w$ and $Q \in \{\forall, \exists\}$ be such that $\Phi \vdash Q(o, p)$ and $\Phi \vdash \overline{Q}(o, c)$. Then we have one of the derivations

$$\frac{\begin{array}{c} \vdots \\ \exists(p, o) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, c) \end{array}}{\exists(p, c)} \text{ (D1)} \qquad \frac{\begin{array}{c} \vdots \\ \exists(o, c) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, p) \end{array}}{\exists(p, c)} \text{ (D2)}.$$

□

Lemma 5.7. *Suppose $u, v, w \in W$ with $(\forall q) \in u$, $(\forall q) \in v$ and $q \in w$. Then $u = v$.*

Proof. By Lemma 5.3, $\Phi \vdash \exists(q, q)$. By (C2), then, $q \in v$. Suppose $c \in u$, where $c \neq \forall q$. (We already know that $(\forall q) \in v$.) By Lemma 5.5, there exists $o \in u$ and $Q \in \{\forall, \exists\}$ such that $\Phi \vdash Q(o, c)$ and $\Phi \vdash \overline{Q}(o, \forall q)$. Thus, we have one of the derivations

$$\frac{\begin{array}{c} \vdots \\ \exists(o, c) \end{array} \quad \begin{array}{c} \vdots \\ \forall(o, \forall q) \end{array}}{\forall(q, c)} \text{ (HH2)} \qquad \frac{\begin{array}{c} \vdots \\ \forall(o, c) \end{array} \quad \begin{array}{c} \vdots \\ \exists(o, \forall q) \end{array}}{\forall(q, c)} \text{ (HH3)},$$

whence, by (C1), $c \in v$. Thus, $u \subseteq v$. The reverse inclusion follows symmetrically. \square

Say that $w \in W$ is *special* if w contains a c-term of the form $\forall q$ such that $\Phi \vdash \exists(q, q)$. Intuitively, special elements are the unique instances of some property q . We now build the structure \mathfrak{A} as follows:

$$\begin{aligned} A &= \{ \langle w, 0 \rangle \mid w \in W \text{ is special} \} \cup \\ &\quad \{ \langle w, i \rangle \mid w \in W \text{ is non-special, } i \in \{-1, 1\} \} \\ p^{\mathfrak{A}} &= \{ \langle w, i \rangle \in A \mid p \in w \}, \text{ for any atom } p. \end{aligned}$$

We remark that, since W is non-empty, A is non-empty; so this construction is legitimate.

Lemma 5.8. *For all elements $a = \langle w, i \rangle \in A$ and all c-terms c , if $c \in w$, then $a \in c^{\mathfrak{A}}$.*

Proof. We consider the possible forms of c in turn.

1. $c = p$ is an atom: The result is immediate by construction of \mathfrak{A} .
2. $c = \bar{p}$: If, also, $p \in w$, Lemma 5.6 guarantees that $\Phi \vdash \exists(p, \bar{p})$, contradicting the supposed consistency of Φ . Hence, $p \notin w$, whence, by the construction of \mathfrak{A} , $a \in (\bar{p})^{\mathfrak{A}}$.
3. $c = \forall p$: Suppose $b = \langle u, j \rangle \in A$ with $b \in p^{\mathfrak{A}}$. By construction of \mathfrak{A} , $p \in u$, so that $\Phi \vdash \exists(p, p)$, by Lemma 5.3. Furthermore, by Lemma 5.2, for some o , $\Phi \vdash \exists(o, \forall p)$, so that we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \exists(o, \forall p) \end{array}}{\forall(p, \forall p)} \text{ (H2)},$$

whence $(\forall p) \in u$ by (C1). By Lemma 5.7, $w = u$, and therefore, by construction of A , $i = j = 0$. Thus, $b \in p^{\mathfrak{A}}$ implies $b = a$, so that $a \in c^{\mathfrak{A}}$.

4. $c = \overline{\forall p}$: Suppose $c \in w$, and assume for the time being that $i \neq 0$. Thus, w is not special. By Lemma 5.2, there exists an atom q such that $\Phi \vdash \exists(q, \overline{\forall p})$, so that we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \exists(q, \overline{\forall p}) \end{array}}{\exists(p, \overline{\forall q})} \text{ (H3)}.$$

Then there exists $w' \in W_0 \subseteq W$ such that $p \in w'$; and, by construction of \mathfrak{A} , there exists $i' \in \{-1, 0, 1\}$ such that both $\langle w', i' \rangle \in q^{\mathfrak{A}}$ and $\langle w', -i' \rangle \in p^{\mathfrak{A}}$. Since $i \neq 0$ we may suppose $i \neq i'$ (transpose i' and $-i'$ if necessary), so that there exists $a' \in A$ with $a' \neq a$ and $a' \in p^{\mathfrak{A}}$. Hence $a \in c^{\mathfrak{A}}$, as required. Now assume $i = 0$. Then w is special, so suppose $(\forall q) \in w$, with $\Phi \vdash \exists(q, q)$. By (C2), $q \in w$, and by Lemma 5.6, $\Phi \vdash \exists(q, \overline{\forall p})$. Again, therefore, by (H3), $\Phi \vdash \exists(p, \overline{\forall q})$. By the construction of W , there exists $w' \in W_0$ such that $p \in w'$ and $\overline{\forall q} \in w'$. By construction of \mathfrak{A} , there exists $i' \in \{-1, 0, 1\}$ such that $\langle w', i' \rangle \in p^{\mathfrak{A}}$. Since $(\forall q) \in w$ and $\overline{\forall q} \in w'$, we know from the final statement of Lemma 5.5 that $w \neq w'$, and therefore $a \neq a'$. Hence $a \in c^{\mathfrak{A}}$, as required. \square

Proof of Theorem 5.1. Let \mathbf{H} be as given above. Soundness of $\vdash_{\mathbf{H}}$ is immediate from the fact each of these rules is valid. For refutation-completeness, let Φ be a set of \mathcal{H} -formulas consistent with respect to $\vdash_{\mathbf{H}}$. If Φ contains no existential formulas, then $\mathfrak{A} \models \Phi$ for any structure \mathfrak{A} in which $p^{\mathfrak{A}} = \emptyset$ for all $p \in \mathbf{P}$. Otherwise, let \mathfrak{A} be constructed as above. It suffices to show that $\mathfrak{A} \models \Phi$. To see this, let $\varphi \in \Phi$. If $\varphi = \exists(p, c)$, then, by construction of W and A , there exist $w \in W$ and $i \in \{-1, 0, 1\}$ such that $p, c \in w$, and $a = \langle w, i \rangle \in A$. By Lemma 5.8, $a \in p^{\mathfrak{A}} \cap c^{\mathfrak{A}}$ so that $\mathfrak{A} \models \varphi$. On the other hand, if $\varphi = \forall(p, c)$, suppose $a = \langle w, i \rangle \in p^{\mathfrak{A}}$. By construction of \mathfrak{A} , $p \in w$, and by Condition (C1), $c \in w'$, whence, by Lemma 5.8, $a \in c^{\mathfrak{A}}$. Thus, $p^{\mathfrak{A}} \subseteq c^{\mathfrak{A}}$, so that $\mathfrak{A} \models \varphi$. \square

6 NPTIME-completeness of \mathcal{H}^{\dagger} and $\mathcal{H}^{*\dagger}$

The objective of this section is to prove

Theorem 6.1. *The problem of determining whether a set of \mathcal{H}^{\dagger} -formulas is satisfiable is NPTIME-complete, and similarly for the problem of determining whether a set of $\mathcal{H}^{*\dagger}$ -formulas is satisfiable.*

From Theorem 6.1 and Proposition 3.2, it follows that, unless $\text{PTIME} = \text{NPTIME}$, there is no finite set \mathbf{X} of syllogistic rules in either \mathcal{H}^{\dagger} or $\mathcal{H}^{*\dagger}$ such that $\vdash_{\mathbf{X}}$ is sound and refutation-complete.

Membership of these problems in NPTIME is easily established by showing that any satisfiable set Φ of $\mathcal{H}^{*\dagger}$ -formulas is satisfied in a structure whose size is bounded by a polynomial function of the number of symbols in Φ . (Alternatively, the same result is an immediate consequence of Theorem 7.2 together with Proposition 3.3.) Therefore, only the lower bounds need be considered. We use a variant of a technique from McAllester and Givan [6]. We remark that

our task would be very easy if we could write a set of \mathcal{H}^\dagger -formulas whose only models have cardinality 3. However, by Theorem 7.11, this is impossible.

The proof of NPTime-hardness proceeds by reduction of the problem 3SAT to the satisfiability problem for \mathcal{H}^\dagger . In this context, a *clause* is an expression $L_1 \vee L_2 \vee L_3$, where each L_k ($1 \leq k \leq 3$) is either a proposition letter o or a negated proposition letter $\neg o$. Given an assignment θ of truth-values (t or f) to proposition letters, any clause γ receives a truth-value $\theta(\gamma)$ in the obvious way. An instance of the problem 3SAT is a set Γ of clauses; that instance is positive just in case there exists a θ such that $\theta(\gamma) = t$ for every $\gamma \in \Gamma$. Let Γ be a finite set of clauses. We show how to compute, in logarithmic space, a set Φ of \mathcal{H}^\dagger -formulas such that Φ is satisfiable if and only if Γ is a positive instance of 3SAT. To make the proof easier, we work first with $\mathcal{H}^{*\dagger}$ -formulas, strengthening the result at the very end of the proof.

First, we need formulas to represent proposition letters. For each proposition letter o occurring in Γ , let o_t and o_f be atoms (elements of \mathbf{P}), and let Φ_o be the set of $\mathcal{H}^{*\dagger}$ -formulas:

$$\forall(\forall o_t, \overline{\forall o_f}) \quad \forall(o_t, \forall o_f) \quad \forall(o_t, \overline{o_f}).$$

Intuitively, if $\mathfrak{A} \models \Phi_o$, we are to interpret the equation $o_t^{\mathfrak{A}} = \emptyset$ as stating that o is true, and $o_f^{\mathfrak{A}} = \emptyset$ as stating that o is false. The following lemma justifies this interpretation. Suppose \mathfrak{A} and \mathfrak{B} are structures and $p \in \mathbf{P}$. We say that \mathfrak{A} and \mathfrak{B} *agree on p* if $p^{\mathfrak{A}} = p^{\mathfrak{B}}$. Note that if $\mathfrak{A} \subseteq \mathfrak{B}$, then \mathfrak{A} and \mathfrak{B} agree on p just in case $p^{\mathfrak{B}} \setminus A = \emptyset$.

Lemma 6.2. *If $\mathfrak{A} \models \Phi_o$, then $o_t^{\mathfrak{A}} = \emptyset$ if and only if $o_f^{\mathfrak{A}} \neq \emptyset$. Conversely, suppose A is a 2-element set, and $v \in \{t, f\}$. There exists a structure \mathfrak{A}_o^v over A such that, if $\mathfrak{B} \supseteq \mathfrak{A}_o^v$ agrees with \mathfrak{A}_o^v on o_t and o_f , then $\mathfrak{B} \models \Phi_o$; furthermore, $(o^v)^{\mathfrak{B}} = \emptyset$.*

Proof. For the first statement, suppose $\mathfrak{A} \models \Phi_o$. From $\forall(o_t, \forall o_f)$ and $\forall(o_t, \overline{o_f})$, it is obvious that we cannot have both $o_t^{\mathfrak{A}} \neq \emptyset$ and $o_f^{\mathfrak{A}} \neq \emptyset$. On the other hand, suppose $o_t^{\mathfrak{A}} = \emptyset$. Then every element satisfies $\forall o_t$, and so some element does, whence, from $\forall(\forall o_t, \overline{\forall o_f})$, that element is distinct from some $o_f^{\mathfrak{A}}$, so that $o_f^{\mathfrak{A}} \neq \emptyset$. For the second statement, let $A = \{a, b\}$. Define the structure \mathfrak{A}_o^t by setting $(o_t)^{\mathfrak{A}_o^t} = \emptyset$ and $(o_f)^{\mathfrak{A}_o^t} = A$; similarly, define the structure \mathfrak{A}_o^f by setting $(o_f)^{\mathfrak{A}_o^f} = \emptyset$ and $(o_t)^{\mathfrak{A}_o^f} = A$. A routine check shows that these structures have the specified properties. \square

Next, we need formulas to represent clauses. For each clause $\gamma = L_1 \vee L_2 \vee L_3 \in \Gamma$, let $s_{\gamma,1}, s_{\gamma,2}, s_{\gamma,3}, s_{\gamma,4}, p_{\gamma,1}, p_{\gamma,2}$ and $p_{\gamma,3}$ be atoms (elements of \mathbf{P}); in addition, let Φ_γ be the set of \mathcal{H}^\dagger -formulas:

$$\begin{aligned} \forall(s_{\gamma,1}, \forall p_{\gamma,1}) & \quad \forall(p_{\gamma,1}, s_{\gamma,2}) \\ \forall(s_{\gamma,2}, \forall p_{\gamma,2}) & \quad \forall(p_{\gamma,2}, s_{\gamma,3}) \\ \forall(s_{\gamma,3}, \forall p_{\gamma,3}) & \quad \forall(p_{\gamma,3}, s_{\gamma,4}) \\ \exists(s_{\gamma,1}, \overline{s_{\gamma,4}}). \end{aligned}$$

Intuitively, if $\mathfrak{A} \models \Phi_\gamma$ we are to interpret the equation $p_{\gamma,k}^{\mathfrak{A}} = \emptyset$ as stating that L_k is true ($1 \leq k \leq 3$). The next lemma justifies this interpretation.

Lemma 6.3. *If $\mathfrak{A} \models \Phi_\gamma$, then the set of numbers k ($1 \leq k \leq 3$) such that $(p_{\gamma,k})^{\mathfrak{A}} = \emptyset$ is non-empty. Conversely, suppose A is a 2-element set, and K a non-empty subset of $\{1, 2, 3\}$. There exists a structure \mathfrak{A}_γ^K over A such that, if $\mathfrak{B} \supseteq \mathfrak{A}_\gamma^K$ agrees with \mathfrak{A}_γ^K on the atoms in $\{s_{\gamma,1}, s_{\gamma,2}, s_{\gamma,3}, s_{\gamma,4}, p_{\gamma,1}, p_{\gamma,2}, p_{\gamma,3}\}$, then $\mathfrak{B} \models \Phi_\gamma$; furthermore, for all k ($1 \leq k \leq 3$), $(p_{\gamma,k})^{\mathfrak{B}} = \emptyset$ if and only if $k \in K$.*

Proof. For the first statement, suppose, for contradiction, that $\mathfrak{A} \models \Phi_\gamma$, but $(p_{\gamma,k})^{\mathfrak{A}} \neq \emptyset$, for all k ($1 \leq k \leq 3$). Since $\mathfrak{A} \models \exists(s_{\gamma,1}, \bar{s}_{\gamma,4})$, let $a \in s_{\gamma,1}^{\mathfrak{A}} \setminus s_{\gamma,4}^{\mathfrak{A}}$. Since $\mathfrak{A} \models \forall(s_{\gamma,1}, \forall p_{\gamma,1})$, and neither $s_{\gamma,1}^{\mathfrak{A}}$ nor $p_{\gamma,1}^{\mathfrak{A}}$ is empty, we have $a \in p_{\gamma,1}^{\mathfrak{A}}$; moreover, since, $\mathfrak{A} \models \forall(p_{\gamma,1}, s_{\gamma,2})$, $a \in s_{\gamma,2}^{\mathfrak{A}} \setminus s_{\gamma,4}^{\mathfrak{A}}$. Repeating the same reasoning twice over, $a \in s_{\gamma,4}^{\mathfrak{A}} \setminus s_{\gamma,4}^{\mathfrak{A}}$, a contradiction.

For the second statement of the lemma, let $A = \{a, b\}$, and define \mathfrak{A}_γ^K according to the following table.

K	atoms satisfied by a	atoms satisfied by b
$\{1\}$	$s_{\gamma,1}$	$p_{\gamma,2}, p_{\gamma,3}, s_{\gamma,3}, s_{\gamma,4}$
$\{2\}$	$p_{\gamma,1}, s_{\gamma,1}, s_{\gamma,2}$	$p_{\gamma,3}, s_{\gamma,4}$
$\{3\}$	$p_{\gamma,1}, p_{\gamma,2}, s_{\gamma,1}, s_{\gamma,2}, s_{\gamma,3}$	-
$\{2, 3\}$	$p_{\gamma,1}, s_{\gamma,1}, s_{\gamma,2}$	-
$\{1, 3\}$	$p_{\gamma,2}, s_{\gamma,1}, s_{\gamma,3}$	-
$\{1, 2\}$	$s_{\gamma,1}$	$p_{\gamma,3}, s_{\gamma,4}$
$\{1, 2, 3\}$	$s_{\gamma,1}$	-

An exhaustive check shows that \mathfrak{A}_γ^K has the required properties. \square

Finally, we need formulas to link proposition letters and clauses. For each clause $\gamma = L_1 \vee L_2 \vee L_3 \in \Gamma$, and for all k ($1 \leq k \leq 3$), let the \mathcal{H}^\dagger -formula $\psi_{\gamma,k}$ be given by

$$\psi_{\gamma,k} = \begin{cases} \forall(o_t, \overline{\forall p_{\gamma,k}}) & \text{if } L_k = o \\ \forall(o_f, \overline{\forall p_{\gamma,k}}) & \text{if } L_k = \neg o, \end{cases}$$

and let $\Psi_\gamma = \{\psi_{\gamma,1}, \psi_{\gamma,2}, \psi_{\gamma,3}\}$.

Lemma 6.4. *Suppose $\mathfrak{A} \models \Psi_\gamma$, and $(p_{\gamma,k})^{\mathfrak{A}} = \emptyset$. If $L_k = o$, then $o_t^{\mathfrak{A}} = \emptyset$; and if $L_k = \neg o$, then $(o_f)^{\mathfrak{A}} = \emptyset$.*

Proof. Immediate. \square

Proof of Theorem 6.1. We need only show NPTime-hardness. To this end, let Γ be a set of clauses over the proposition letters occurring in Γ . Let

$$\Phi = \bigcup \{\Phi_o \mid o \text{ occurs in } \Gamma\} \cup \bigcup \{\Phi_\gamma \cup \Psi_\gamma \mid \gamma \in \Gamma\}.$$

We claim that Φ is satisfiable if and only if Γ is. For suppose $\mathfrak{A} \models \Phi$. Define the truth-value assignment θ over the proposition letters of Γ by setting $\theta(o) = t$

just in case $o_t^{\mathfrak{A}} = \emptyset$. It follows from Lemma 6.2 that, if o is any proposition letter mentioned in Γ , then $\theta(o) = f$ just in case $o_f^{\mathfrak{A}} = \emptyset$. Now let $\gamma = L_1 \vee L_2 \vee L_3$ be a clause in Γ . By Lemma 6.3, for all $\gamma \in \Gamma$, there exists a k ($1 \leq k \leq 3$) such that $p_{\gamma,k}^{\mathfrak{A}} = \emptyset$. By Lemma 6.4: if $L_k = o$, then $o_t^{\mathfrak{A}} = \emptyset$, so that $\theta(\gamma) = \theta(o) = t$; and if $L_k = \neg o$, then $o_f^{\mathfrak{A}} = \emptyset$, so that $\theta(\gamma) = \theta(\neg o) = t$. Either way, $\theta(\gamma) = t$.

Conversely, suppose θ is a truth-value assignment such that $\theta(\gamma) = t$ for all $\gamma \in \Gamma$. For all o occurring in Γ , let \mathfrak{A}_o be the structure $\mathfrak{A}_o^{\theta(o)}$ over domain A_o guaranteed by Lemma 6.2. For each $\gamma = L_1 \vee L_2 \vee L_3 \in \Gamma$, the set $K = \{k \mid 1 \leq k \leq 3 \text{ and } \theta(L_k) = t\}$ is non-empty; so let \mathfrak{A}_γ be the structure \mathfrak{A}_γ^K over domain A_γ guaranteed by Lemma 6.3. Assume the domains of all these structures are disjoint, and let

$$\mathfrak{B} = \bigcup \{\mathfrak{A}_o \mid o \text{ occurs in } \Gamma\} \cup \bigcup \{\mathfrak{A}_\gamma \mid \gamma \in \Gamma\}.$$

Thus, for all o occurring in Γ , \mathfrak{B} agrees with \mathfrak{A}_o on the atoms o_t and o_f , whence $\mathfrak{B} \models \Phi_o$. Likewise, for all $\gamma \in \Gamma$, \mathfrak{B} agrees with \mathfrak{A}_γ on the atoms in $\{s_{\gamma,1}, s_{\gamma,2}, s_{\gamma,3}, s_{\gamma,4}, p_{\gamma,1}, p_{\gamma,2}, p_{\gamma,3}\}$, whence $\mathfrak{B} \models \Phi_\gamma$. It remains to show that $\mathfrak{B} \models \Psi_\gamma$ for each $\gamma \in \Gamma$. Suppose $\gamma = L_1 \vee L_2 \vee L_3 \in \Gamma$, and $1 \leq k \leq 3$. If $L_k = o$, then $\psi_{\gamma,k} = \forall(o_t, \overline{\forall p_{\gamma,k}})$. Take any $a \in o_t^{\mathfrak{B}}$. By the construction of \mathfrak{B} , $a \in A_o$, and $\theta(o) = f$, whence $\theta(L_k) = f$, whence, by the construction of \mathfrak{B} again, $p_{\gamma,k}^{\mathfrak{B}} \cap A_\gamma \neq \emptyset$. Since A_o and A_γ are disjoint, $\mathfrak{B} \models \psi_{\gamma,k}$. On the other hand, if $L_k = \neg o$, then $\psi_{\gamma,k} = \forall(o_f, \overline{\forall p_{\gamma,k}})$. Take any $a \in o_f^{\mathfrak{B}}$. By the construction of \mathfrak{B} , $a \in A_o$, and $\theta(o) = t$, whence $\theta(L_k) = f$, whence, by the construction of \mathfrak{B} again, $p_{\gamma,k}^{\mathfrak{B}} \cap A_\gamma \neq \emptyset$. Since A_o and A_γ are disjoint, we again have $\mathfrak{B} \models \psi_{\gamma,k}$. Thus, $\mathfrak{B} \models \Phi$. This establishes the NP-TIME-hardness of the satisfiability problem for $\mathcal{H}^{*\dagger}$.

To extend the result to \mathcal{H}^\dagger , note that the only formulas of Φ not in \mathcal{H}^\dagger are those the forms $\forall(\forall o_t, \overline{\forall o_f})$ occurring in Φ_o . But we can simply replace any such formula, equisatisfiably, by the pair of formulas $\forall(q, \overline{\forall o_f}), \forall(\bar{q}, \overline{\forall o_t})$, where q is a fresh atom. \square

7 Complete indirect syllogistic systems for \mathcal{H}^\dagger and $\mathcal{H}^{*\dagger}$

The objective of this section is to prove

Theorem 7.1. *There is a finite set \mathbf{H}^\dagger of syllogistic rules in \mathcal{H}^\dagger such that the indirect derivation relation $\vdash_{\mathbf{H}^\dagger}$ is sound and complete.*

Theorem 7.2. *There is a finite set $\mathbf{H}^{*\dagger}$ of syllogistic rules in $\mathcal{H}^{*\dagger}$ such that the indirect derivation relation $\vdash_{\mathbf{H}^{*\dagger}}$ is sound and complete.*

We present first the proof of Theorem 7.1. The proof of Theorem 7.2 proceeds similarly (and in fact more simply); we indicate merely the differences between the two proofs.

Let \mathbf{H}^\dagger consist of the following rules:

1. ‘little’ rules:

$$\begin{array}{ll} \frac{\exists(\ell, c)}{\exists(\ell, \ell)} \text{ (I)} & \frac{}{\forall(\ell, \ell)} \text{ (T)} \\[1em] \frac{\forall(c, \ell) \quad \forall(c, \bar{\ell})}{\forall(c, m)} \text{ (A)} & \frac{\forall(\ell, \bar{\ell})}{\exists(\ell, \ell)} \text{ (N);} \end{array}$$

2. generalizations of classical syllogisms:

$$\begin{array}{ll} \frac{\forall(\ell, m) \quad \forall(m, c)}{\forall(\ell, c)} \text{ (B1)} & \frac{\forall(\ell, c) \quad \forall(c, m)}{\forall(\ell, m)} \text{ (B2)} \\[1em] \frac{\exists(\ell, m) \quad \forall(m, c)}{\exists(\ell, c)} \text{ (D1)} & \frac{\exists(\ell, c) \quad \forall(c, m)}{\exists(\ell, m)} \text{ (D2);} \end{array}$$

3. the ‘Hamiltonian’ rules:

$$\begin{array}{ll} \frac{\exists(\ell, c) \quad \exists(m, \forall \ell)}{\forall(\ell, c)} \text{ (HH1)} & \frac{\exists(m, \forall \ell)}{\forall(\ell, \forall \ell)} \text{ (H2)} \\[1em] \frac{\exists(\ell, \forall m)}{\exists(m, \forall \ell)} \text{ (H3)} & \frac{\exists(\ell, \ell)}{\forall(\bar{\ell}, \forall \bar{\ell})} \text{ (H4).} \end{array}$$

To avoid unnecessary proliferation of rule-names, those rules which are simple generalizations of rules in \mathbf{H} have been given the same names. Again, establishing the validity of the rules in \mathbf{H}^\dagger is straightforward. Rule (A) is valid because its premises imply that nothing is a c ; we cannot replace (A) with the simpler schema $\forall(c, \bar{c})/\forall(c, m)$, because, if c is not a literal, $\forall(c, \bar{c})$ is not in the language \mathcal{H}^\dagger . Rule (N)—no analogue of which can be formulated in the language \mathcal{H} —is valid because of the assumption that domains are non-empty: if no ℓ s are $\bar{\ell}$ s, then everything is a non- $\bar{\ell}$, and so something is a non- $\bar{\ell}$. Rule (T) can in fact be viewed as a special case of the rule (RAA), since we have the derivation

$$\frac{[\exists(\ell, \bar{\ell})]^1}{\forall(\ell, \ell)} \text{ (RAA)}^1.$$

But we retain (T) as a separate rule for clarity.

Let Φ be a complete set of \mathcal{H}^\dagger -formulas such that Φ is consistent with respect to $\vdash_{\mathbf{H}^\dagger}$. In the following lemmas, we build a structure \mathfrak{A} , and show that $\mathfrak{A} \models \Phi$. Since \mathbf{H}^\dagger is the only set of rules we shall be concerned with in the ensuing lemmas, until further notice we write \vdash for the direct proof-relation $\vdash_{\mathbf{H}^\dagger}$.

The elements of A are constructed using sets of c-terms. Call a set S of c-terms *consistent* if, for every c-term c , $c \in S$ implies $\bar{c} \notin S$, and *literal-complete* if, for every literal ℓ , $\ell \notin S$ implies $\bar{\ell} \in S$. Notice that the notion of consistency for sets of c-terms is not the same as \vdash -consistency for sets of formulas; likewise,

literal-completeness for sets of c-terms is not the same as completeness for sets of formulas. Let S be any set of c-terms. Define

$$S^* = S \cup \{c \mid \text{there exists } \ell \in S \text{ such that } \Phi \vdash \forall(\ell, c)\} \cup \{\ell \mid \text{there exists } c \in S \text{ such that } \Phi \vdash \forall(c, \ell)\}.$$

and we call S *closed* if $S = S^*$. Trivially, $S \subseteq S^*$.

Lemma 7.3. *Let S be a set of c-terms. Then S^* is closed.*

Proof. We suppose $d \in (S^*)^* \setminus S^*$, and derive a contradiction. We consider first the case where $d = m$ is a literal. By definition, there exists $c \in S^*$ such that $\Phi \vdash \forall(c, m)$. Certainly, $c \notin S$, for otherwise, we would have $d \in S^*$. Suppose first that c is not a literal. Then there exists $\ell \in S$ such that $\Phi \vdash \forall(\ell, c)$, and we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \forall(\ell, c) \end{array} \quad \begin{array}{c} \vdots \\ \forall(c, m) \end{array}}{\forall(\ell, m)} \text{ (B2)},$$

so that $m \in S^*$, a contradiction. On the other hand, suppose $c = \ell$ is a literal. Then there exists a c-term $c_0 \in S$ such that $\Phi \vdash \forall(c_0, \ell)$. Taking account of the equivalence of $\forall(e, f)$ and $\forall(\bar{f}, \bar{e})$, we have the derivation

$$\frac{\begin{array}{c} \vdots \\ \forall(\bar{m}, \bar{\ell}) \end{array} \quad \begin{array}{c} \vdots \\ \forall(\bar{\ell}, \bar{c}_0) \end{array}}{\forall(\bar{m}, \bar{c}_0)} \text{ (B1)},$$

i.e. $\Phi \vdash \forall(c_0, m)$, so that $m \in S^*$, a contradiction. The case where d is not a literal proceeds similarly (in fact, more simply). \square

Lemma 7.4. *Every closed, consistent set of c-terms containing at least one literal has a closed, consistent, literal-complete extension.*

Proof. Enumerate the literals as ℓ_0, ℓ_1, \dots , and suppose S is closed and consistent. Define $S^{(0)} = S$, and

$$S^{(i+1)} = \begin{cases} (S^{(i)} \cup \{\ell_i\})^* & \text{if } \bar{\ell}_i \notin S^{(i)} \\ S^{(i)} & \text{otherwise,} \end{cases}$$

for all $i \geq 0$. It follows from Lemma 7.3 that each $S^{(i)}$ is closed; we show by induction that it is also consistent. From this it follows that $\bigcup_{0 \leq i} S^{(i)}$ is consistent, thus proving the lemma. The case $i = 0$ is true by hypothesis; so we suppose that $S^{(i)}$ is consistent, but $S^{(i+1)}$ inconsistent, and derive a contradiction. Let m_0 be a literal in $S^{(0)}$, and hence in $S^{(i)}$; and let c be a c-term such that $c, \bar{c} \in S^{(i+1)}$. Since $S^{(i)}$ is consistent, by exchanging c and \bar{c} if necessary, we may assume that $c \notin S^{(i)}$. And since $S^{(i)}$ is also closed, we know that either $c = \ell_i$ or $\Phi \vdash \forall(\ell_i, c)$. Indeed, by rule (T), the latter

case subsumes the former. Therefore, $\bar{c} \notin S^{(i)}$, since, otherwise, we would have $d = \bar{c} \in S^{(i)}$ such that $\Phi \vdash \forall(d, \bar{\ell}_i)$, whence $\bar{\ell}_i \in S^{(i)}$, contrary to assumption. Since $\bar{c} \in S^{(i+1)}$, it follows—again taking account of rule (T)—that $\Phi \vdash \forall(\ell_i, \bar{c})$. But then we have the derivation

$$\frac{\frac{\vdots}{\forall(\ell_i, \ell_i)} \text{ (T)} \quad \frac{\frac{\vdots}{\forall(\ell_i, c)} \quad \frac{\vdots}{\forall(c, \bar{\ell}_i)} \text{ (B2)}}{\forall(\ell_i, \bar{\ell}_i)} \text{ (A)}}{\forall(\ell_i, \bar{m}_0)} \text{ (A),}$$

so that $\Phi \vdash \forall(m_0, \bar{\ell}_i)$, again contrary to the fact that $S^{(i)} \neq S^{(i+1)}$. \square

Denote by W the set of all closed, consistent and literal-complete sets of c-terms. In the sequel, we use the variables u, v, w to range over W .

Lemma 7.5. *Suppose $\Phi \vdash \exists(\ell, c)$. Then there exists $w \in W$ such that $\ell, c \in w$.*

Proof. By Lemmas 7.3 and 7.4, we need only show that $\{\ell, c\}^*$ is consistent. So suppose otherwise. Since Φ is \vdash -consistent, $c \neq \bar{\ell}$. We therefore have the following possible cases: (i) $\Phi \vdash \forall(\ell, \bar{c})$; (ii) there exists d such that $\Phi \vdash \forall(\ell, d)$ and $\Phi \vdash \forall(\ell, \bar{d})$; (iii) there exists d such that $\Phi \vdash \forall(\ell, d)$ and $\Phi \vdash \forall(c, \bar{d})$; (iv) there exists d such that $\Phi \vdash \forall(c, d)$ and $\Phi \vdash \forall(c, \bar{d})$. Note that, in Cases (iii) and (iv), one of c or d must be a literal. In Case (i), Rule (D2) immediately yields $\Phi \vdash \exists(\ell, \bar{\ell})$. In Case (ii), we have the derivation:

$$\frac{\frac{\frac{\vdots}{\exists(\ell, c)} \quad \frac{\vdots}{\exists(\ell, \ell)} \text{ (I)}}{\exists(\ell, \bar{d})} \text{ (D1)} \quad \frac{\frac{\vdots}{\forall(\ell, \bar{d})} \quad \frac{\vdots}{\forall(\bar{d}, \bar{\ell})} \text{ (D2)}}{\exists(\ell, \bar{\ell})} \text{ (D2).}$$

Likewise, in case (iii), we have the derivation:

$$\frac{\frac{\frac{\vdots}{\exists(\ell, c)} \quad \frac{\vdots}{\forall(c, \bar{d})} \text{ (D1) or (D2)}}{\exists(\ell, \bar{d})} \quad \frac{\frac{\vdots}{\forall(\bar{d}, \bar{\ell})} \text{ (D1) or (D2)}}{\exists(\ell, \bar{\ell})} \text{ (D1) or (D2).}$$

In Case (iv), if c is a literal, we proceed as in Case (ii), but with ℓ and c exchanged; and if $d = m$ is a literal, we have the derivation:

$$\frac{\frac{\frac{\vdots}{\exists(\ell, c)} \quad \frac{\frac{\vdots}{\forall(c, m)} \quad \frac{\vdots}{\forall(c, \bar{m})} \text{ (A)}}{\forall(c, \bar{\ell})} \text{ (D2)}}{\exists(\ell, \bar{\ell})}$$

Since all cases contradict the supposed \vdash -consistency of Φ , the lemma is proved. \square

Lemma 7.6. *The set W is not empty.*

Proof. By Lemma 7.5, it is necessary only to show that $\Phi \vdash \exists(\ell, c)$ for some ℓ and c . Pick any ℓ . If $\exists(\ell, \ell) \in \Phi$, we are done. Otherwise, by completeness of Φ , $\forall(\ell, \bar{\ell}) \in \Phi$, so that, by Rule (N), $\Phi \vdash \exists(\bar{\ell}, \bar{\ell})$, completing the proof. \square

The following lemma is the analogue, for the system H^\dagger , of Lemma 5.6. This time, however, the lemma is trivial, because we are assuming that Φ is complete.

Lemma 7.7. *Suppose $\ell, c \in w \in W$. Then $\exists(\ell, c) \in \Phi$.*

Proof. Suppose $\exists(\ell, c) \notin \Phi$. By the completeness of Φ , $\forall(\ell, \bar{c}) \in \Phi$, whence $\bar{c} \in w$, because w is closed. This contradicts the consistency of w . \square

Lemma 7.8. *Suppose $w \in W$, and $(\forall \ell) \in w$, where $\Phi \vdash \exists(\ell, \ell)$. Then $\ell \in w$.*

Proof. Suppose otherwise. By the literal-completeness of w , $\bar{\ell} \in w$. But we have the derivation

$$\frac{\exists(\ell, \ell)}{\forall(\bar{\ell}, \bar{\forall \ell})} \text{ (H4),}$$

so that, since w is closed, $\bar{\forall \ell} \in w$, contradicting the consistency of w . \square

Lemma 7.9. *Suppose $u, v, w \in W$ with $(\forall \ell) \in u$, $(\forall \ell) \in v$ and $\ell \in w$. Then $u = v$.*

Proof. By Lemma 7.7, $\Phi \models \exists(\ell, \ell)$. By Lemma 7.8, $\ell \in w$ and $\ell \in v$. Suppose also $c \in u$. By Lemma 7.7 again, $\exists(\ell, c) \in \Phi$, and $\exists(\ell, \forall \ell) \in \Phi$. Therefore, we have the derivation

$$\frac{\exists(\ell, c) \quad \exists(\ell, \forall \ell)}{\forall(\ell, c)} \text{ (HH1),}$$

and $c \in v$. Thus, $u \subseteq v$. The reverse inclusion follows symmetrically. \square

Analogously to Section 5, we call $w \in W$ *special* if it contains a c-term of the form $\forall \ell$ such that $\Phi \vdash \exists(\ell, \ell)$; and we build the structure \mathfrak{A} as follows:

$$\begin{aligned} A &= \{\langle w, 0 \rangle \mid w \in W \text{ is special}\} \cup \\ &\quad \{\langle w, i \rangle \mid w \in W \text{ is non-special}, i \in \{-1, 1\}\} \\ p^{\mathfrak{A}} &= \{\langle w, i \rangle \in A \mid p \in w\}, \text{ for any atom } p. \end{aligned}$$

We remark that, since, by Lemma 7.6, W is non-empty, A is non-empty; so this construction is legitimate.

Lemma 7.10. *Suppose c is a c-term and $a = \langle w, i \rangle$. Then $c \in w$ implies $a \in c^{\mathfrak{A}}$. Further, if ℓ is a literal, Then $a \in \ell^{\mathfrak{A}}$ implies $\ell \in w$.*

Proof. We consider the possible forms of c in turn.

1. $c = p$ is an atom: By construction of \mathfrak{A} , $c \in w$ if and only if $p \in w$.
2. $c = \bar{p}$: By consistency and literal-completeness of w , $c \in w$ if and only if $p \notin w$. The result then follows by Case 1.
3. $c = \forall \ell$: Suppose $c \in w$, and $a' = \langle w', i' \rangle$ is such that $a' \in \ell^{\mathfrak{A}}$. By Cases 1 and 2, $\ell \in w'$. Pick any literal $m \in w$. By Lemma 7.7, $\exists(\ell, \ell) \in \Phi$ and $\exists(m, \forall \ell) \in \Phi$. Thus, we have the derivation

$$\frac{\exists(m, \forall \ell)}{\forall(\ell, \forall \ell)} \text{ (H2),}$$

whence $\forall \ell \in w'$, and therefore, by Lemma 7.9, $w = w'$. Indeed, since w is special, the construction of A ensures that $i = i' = 0$, and hence $a = a'$. Thus, $a' \in \ell^{\mathfrak{A}}$ implies $a = a'$, whence $a \in c^{\mathfrak{A}}$, as required.

4. $c = \overline{\forall \ell}$: Suppose $c \in w$, and assume for the time being that $i \neq 0$. Pick any literal $m \in w$. By Lemma 7.7, $\exists(m, \overline{\forall \ell}) \in \Phi$, so that we have the derivation

$$\frac{\exists(m, \overline{\forall \ell})}{\exists(\ell, \overline{\forall m})} \text{ (H3).}$$

By Lemma 7.5, there exists $w' \in W$ such that $\ell \in w'$, and by construction of A and Cases 1 and 2 above, there exists $i' \in \{-1, 0, 1\}$ such that both $\langle w', i' \rangle \in \ell^{\mathfrak{A}}$ and $\langle w', -i' \rangle \in \ell^{\mathfrak{A}}$. Since $i \neq 0$ we may suppose $i \neq i'$, so that there exists $a' \in A$ with $a' \neq a$ and $a' \in \ell^{\mathfrak{A}}$. Hence $a \in c^{\mathfrak{A}}$, as required. Now assume $i = 0$. Then w is special, so suppose $(\forall m) \in w$, with $\Phi \vdash \exists(m, m)$. By Lemma 7.8, $m \in w$, so that, by Lemma 7.7, $\exists(m, \overline{\forall \ell}) \in \Phi$. Again, then, by (H3) and Lemma 7.5, there exists $w' \in W$ such that $\ell \in w'$ and also $\overline{\forall m} \in w'$. By construction of \mathfrak{A} and Cases 1 and 2 above, there exists $i' \in \{-1, 0, 1\}$ such that $\langle w', i' \rangle \in \ell^{\mathfrak{A}}$. Since $\forall m \in w$, we have $w \neq w'$, and therefore $a \neq a'$. Hence $a \in c^{\mathfrak{A}}$, as required. \square

Proof of Theorem 7.2. Since we are dealing with an indirect proof relation, it suffices to show that every $\models_{\mathcal{H}^+}$ -consistent set of formulas is true in some structure. Let Φ be $\models_{\mathcal{H}^+}$ -consistent. By Lemma 3.4, we may further assume without loss of generality that Φ is complete. Certainly, Φ is $\vdash_{\mathcal{H}^+}$ -consistent. Let \mathfrak{A} be constructed as described above: we show that $\mathfrak{A} \models \Phi$. For suppose $\varphi = \exists(\ell, c) \in \Phi$. By Lemma 7.5, there exists $a = \langle w, i \rangle \in A$ such that $\ell, c \in w$. By Lemma 7.10, $a \in \ell^{\mathfrak{A}}$ and $a \in c^{\mathfrak{A}}$; thus, $\mathfrak{A} \models \varphi$. On the other hand, suppose $\varphi = \forall(\ell, c) \in \Phi$. If $a = \langle w, i \rangle \in \ell^{\mathfrak{A}}$, then, by (the second statement of) Lemma 7.10, $\ell \in w$, whence, by the fact that w is closed, $c \in w$, whence $a \in c^{\mathfrak{A}}$, by Lemma 7.10; thus, $\mathfrak{A} \models \varphi$. \square

Turning now to the language $\mathcal{H}^{*\dagger}$, let $\mathcal{H}^{*\dagger}$ consist of the following rules:

1. ‘little’ rules:

$$\frac{\exists(e, f)}{\exists(e, e)} \text{ (I)} \quad \frac{}{\forall(e, e)} \text{ (T)} \quad \frac{\forall(e, \bar{e})}{\forall(f, \bar{e})} \text{ (A)} \quad \frac{\forall(e, \bar{e})}{\exists(\bar{e}, \bar{e})} \text{ (N)};$$

2. generalizations of classical syllogisms:

$$\frac{\forall(e, f) \quad \forall(f, g)}{\forall(e, g)} \text{ (B)} \quad \frac{\exists(e, f) \quad \forall(f, g)}{\exists(e, g)} \text{ (D)};$$

3. the ‘Hamiltonian’ rules:

$$\begin{array}{ll} \frac{\exists(\ell, e) \quad \exists(m, \forall \ell)}{\forall(\ell, e)} \text{ (HH1)} & \frac{\exists(e, \forall \ell)}{\forall(\ell, \forall \ell)} \text{ (H2)} \\ \frac{\exists(\ell, \overline{\forall m})}{\exists(m, \overline{\forall \ell})} \text{ (H3)} & \frac{\exists(\ell, \ell)}{\forall(\bar{\ell}, \overline{\forall \ell})} \text{ (H4)}. \end{array}$$

Where rules in $\mathbf{H}^{*\dagger}$ are obvious generalizations of counterparts in \mathbf{H}^\dagger , we have kept the same names. Otherwise, $\mathbf{H}^{*\dagger}$ is simpler than \mathbf{H}^\dagger : in particular, Rule (A) now has only one premise, and Rules (B1) and (B2) have been subsumed under the more general Rule (B); similarly for (D1) and (D2). The proof that $\Vdash_{\mathbf{H}^{*\dagger}}$ is complete for $\mathcal{H}^{*\dagger}$ proceeds as for Theorem 7.1, the essential difference being that various complications arising from the restricted syntax of \mathbf{H}^\dagger disappear. Consequently, we confine ourselves to a proof sketch.

Let Φ be a complete set of $\mathcal{H}^{*\dagger}$ -formulas such that Φ is $\vdash_{\mathbf{H}^{*\dagger}}$ -consistent. We build a structure \mathfrak{A} , and show that $\mathfrak{A} \models \Phi$, this time writing \vdash to mean $\vdash_{\mathbf{H}^{*\dagger}}$. The elements of A are constructed using sets of e-terms. Call a set of e-terms S *consistent* if, for every e-term e , $e \in S$ implies $\bar{e} \notin S$, and *term-complete* if, for every e-term e , $e \notin S$ implies $\bar{e} \in S$. If S is a set of e-terms, define

$$S^* = \{f \mid \text{there exists } e \in S \text{ such that } \Phi \vdash \forall(e, f)\},$$

and we call S *closed* if $S = S^*$. Note that the definition of S^* for the system $\mathbf{H}^{*\dagger}$ is simpler than the corresponding definition for \mathbf{H}^\dagger . For any set S of e-terms, it is immediate from Rule (T) that $S \subseteq S^*$, and immediate from Rule (B) that S^* is closed (the analogue of Lemma 7.3). We now define W to be the set of all closed, consistent and term-complete sets of e-terms; and we show, analogously to Lemmas 7.5–7.7, that W is non-empty, and that, for any e-terms e and f , $\exists(e, f) \in \Phi$ if and only if there exists $w \in W$ such that $e, f \in w$. Further, by (HH1) and (H4), we easily show, analogously to Lemma 7.9, that if $u, v, w \in W$ with $(\forall \ell) \in u$, $(\forall \ell) \in v$ and $\ell \in w$, then $u = v$. Defining

$$\begin{aligned} A &= \{\langle w, 0 \rangle \mid w \in W \text{ is special}\} \cup \\ &\quad \{\langle w, i \rangle \mid w \in W \text{ is non-special, } i \in \{-1, 1\}\} \\ p^{\mathfrak{A}} &= \{\langle w, i \rangle \in A \mid p \in w\}, \text{ for any atom } p, \end{aligned}$$

we show, analogously to Lemma 7.10, that, for any e-term e and any domain element $a = \langle w, i \rangle$, $a \in e^{\mathfrak{A}}$ if and only if $e \in w$. Note that this is a stronger statement than Lemma 7.10, and uses the fact that w is term-complete, not just literal-complete. The remainder of the argument then proceeds as for Theorem 7.1, but exploiting the fact that, if e is an arbitrary e-term (not just a literal) and $\langle a, i \rangle \in e^{\mathfrak{A}}$, then $e \in w$.

We finish with a proof of the claim made, in passing, at the end of Section 2, regarding models of sets of $\mathcal{H}^{*\dagger}$ -formulas.

Theorem 7.11. *Let Φ be a set of $\mathcal{H}^{*\dagger}$ -formulas. If Φ has a model with three or more elements, then it has arbitrarily large models.*

Proof. Again, we may assume without loss of generality that Φ is a complete set of formulas. If S is a set of e-terms, we use the notation S^* in the sense of the above sketch proof of Theorem 7.2. Suppose $\mathfrak{B} \models \Phi$, with $\omega > |B| \geq 3$. Write $B = \{b_1, \dots, b_n\}$. We may assume that each b_i is the unique element satisfying some literal ℓ_i , since, otherwise, we can add as many duplicate copies of b_i to \mathfrak{B} as we like without affecting the truth of any $\mathcal{H}^{*\dagger}$ -formulas. It follows that, for all i ($1 \leq i \leq n$), the set of e-terms $\{\bar{\ell}_1, \dots, \bar{\ell}_{i-1}, \ell_i, \bar{\ell}_{i+1}, \dots, \bar{\ell}_n\}^*$ is consistent. We claim that $\{\bar{\ell}_1, \dots, \bar{\ell}_n\}^*$ is also consistent. For otherwise, it is immediate from rule (B) that, for some j, k ($1 \leq j \leq k \leq n$), $\Phi \vdash_{\mathcal{H}^{*\dagger}} \forall(\bar{\ell}_j, \ell_k)$, contradicting the assumption that b_k is the unique element of \mathfrak{B} satisfying ℓ_k (remember that $n \geq 3$). Now let \mathfrak{A} be the model constructed in the proof of Theorem 7.2. Since $\{\bar{\ell}_1, \dots, \bar{\ell}_n\}^*$ is consistent, it has a consistent complete extension, w , so that \mathfrak{A} contains some element $a = \langle w, h \rangle$ satisfying $\bar{\ell}_1, \dots, \bar{\ell}_n$. But, by the same token, \mathfrak{A} also contains an element a_i satisfying ℓ_j if and only if $i = j$. Thus, \mathfrak{A} has cardinality at least $n + 1$. \square

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